Derivatives and Minimization

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1 Functions of one variable.

1.1 Review definitions.

Recall the derivative is the slope of a line that is tangent to a curve. Suppose we have a function

$$y = f(x)$$

Suppose that you begin at a point, x_0 (pronounced “x naught”). And you go a little bit up the X axis to a point x. The change is $\Delta x = x - x_0$. The value of y changes as a result, from $f(x_0)$ to $f(x)$. The change is referred to as $\Delta y = f(x) - f(x_0)$.

The derivative is defined as the ratio of Δy to Δx when Δx shrinks to 0.

I’m leaving room here for you to draw a picture:

People use many kinds of notation for the derivative, but the most commonly used are

$$\frac{dy}{dx}$$

or

$$f'(x)$$

or

$$D(f(x))$$

or

$$\dot{y}$$

You get the idea, right? For any thing you read, it may be the author makes up notation. By far the most common are the first two.

1.2 Maxima and Minima

If we want to find the maximum point of a continuous function, we look for the spot at which the derivative equals 0. Again, make a picture.

1.3 The Second Derivative

What if you think of the derivative as a funciton of x. You can make a plot of the derivative in the vicinity of a maximum or minimum, right?

Now take the derivative of the derivative. Various notations for that:

$$\frac{d^2y}{dx^2} \tag{1}$$

or

$$f''(x)$$

or

$$D^2(f(x))$$

The second derivative is the change in the change. Humph!

Think of the first derivative like “velocity”.

The second derivative is the “acceleration” or “decelleration”.

If a curve is straight, the derivative is the slope and the second derivative would be 0 because slope is not changing.

1.4 Second order conditions.

If you find the value of x for which $f'(x) = 0$, then you are either at a maximum or a minimum. (I mean “locally”, in the immediate vicinity of x.)

If f”(x) >0, then you are at a minimum. The curve f is “concave up.”

if f”(x) < 0, then you are at a maximum. The curve f is “concave down.”

2 Functions of several variables.

2.1 Functions of several variables.

Suppose you have 3 input variables, x_1, x_2, and x_3.

$$y = f(x_1, x_2, x_3)$$

The calculus of many variables can get really complicated, but most of the time it is really simple.

A **partial derivative** is the change in $f(x_1, x_2, x_3)$that results when all of the variables are being held constant except one. The most common notation for the partial derivative is

$$\frac{\partial y}{\partial x_1} \tag{2}$$

Because it is tedious to type that fraction all of the time, you sometimes see authors inventing convenient notation for partial derivatives. My personal favorite is to use a subscript to tell which variable I'm allowing to change:

$$f_1(x_1, x_2, x_3)$$

That is supposed to be the same as 2

2.2 Finding Optima

If you are given a function like $f(x_1, x_2, x_3)$ and you are instructed to find the maximum or mimimum, the first thing you do is find the place where ALL OF THE PARTIAL DERIVATIVES equal 0. That is, solve this system of equations:

$$\frac{\partial y}{\partial x_1} = 0$$

$$\frac{\partial y}{\partial x_2} = 0$$

$$\frac{\partial y}{\partial x_3} = 0$$

Note, you could as well think of this as a matrix of derivatives:

$$D = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \frac{\partial y}{\partial x_3} \end{bmatrix} = 0 \tag{3}$$

One tires quickly of writing down 3 rows of derivatives over and over, so one often just refers to this condition for a maximum or minimum as $D = 0$.

2.3 Second order conditions

In the calculus of one variable, it is easy to tell if one has found a maximum or a minimum from the second derivative.

It is not so simple in the calculus of several variables. It is easy to calculate a second partial derivative of $f()$with respect to x_1

$$\frac{\partial^2 y}{\partial x_1 \partial x_1} = \frac{\partial^2 y}{\partial^2 x_1}$$

And one can also find the partial of $\frac{\partial y}{\partial x_1}$with respect to another variable, say x_2.

$$\frac{\partial^2 y}{\partial x_1 \partial x_2}$$

I prefer short hand notation like $f_{11}()$ or $f_{12}()$ for these.

Anyway, suppose you begin with the matrix of first partials. You can differentiate each item by each of the 3 variables, so that means you can build up a 3x3 matrix of second partial derivatives like so:

$$D' = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1 \partial x_1} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_1 \partial x_3} \\ \frac{\partial^2 y}{\partial x_1 \partial x_2} & \frac{\partial^2 y}{\partial x_2 \partial x_2} & \frac{\partial^2 y}{\partial x_2 \partial x_3} \\ \frac{\partial^2 y}{\partial x_3 \partial x_1} & \frac{\partial^2 y}{\partial x_3 \partial x_2} & \frac{\partial^2 y}{\partial x_3 \partial x_3} \end{bmatrix} \tag{4}$$

This is the so-called **Hessian matrix.** There are various conditions that can be set so that one can diagnose the question of whether a maximum, a minimum, or neither, has been found.

The one that sticks in my mind is the idea of a “positive definite” matrix. Take a $3x1$ column vector z. It is supposed to represent a small change from the location where one currently is, (x_1, x_2, x_3). Calculate the quantity:

$$z' \cdot D' \cdot z \tag{5}$$

or, more verbosely,

$$[z_1, z_2, z_3] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{6}$$

There’s a theorem that says:

If z’D’z > 0, then D’ is a positive definite matrix.

If z’D’z < 0, then D’ is a negative definite matrix.

Use these ideas to check if you have found a maximum or a minimum. Find the point where all the partials are 0, call it $x*$ and then evaluate the Hessian at that point.

If the Hessian is positive definite, you have found a minimum point.

If the Hessian is negative definite, you have found a maximum point.

Note this is entirely similar to the univariate case, where $f''(x) > 0$ means you have a minimum and $f''(x) < 0$ means you have a maximum.