# Linear operators

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## 1 Linear Operator

"Linear operator" sounds jargonish. But it is a pretty important term. In just about any problem in statistics, the linearity of an operator will make for massive simplification.

### 1.1 Definition

A linear operator is one that can "take in" a sum and give back a result in the form of a sum of the applied operators. That is, it is like $F()$ in an expression like this

$$F(x+y+z) = F(x) + F(y) + F(z)$$

Here you see that a linear operator has a distributive quality.

In my usual poetic style,

> The operator over a sum is the sum of the applications of the operator.

If you throw in constants $a$, $b$, and $c$, then linearity also means that

$$F(ax+by+cz) = aF(x) + bF(y) + cF(z)$$

### 1.2 Linear Operators you already know and love.

You already know many linear operators.

#### 1.2.1 Summation.

This is obvious, isn't it? This is just the principle of addition.

$$\sum(x+y) = \sum(x) + \sum(y).$$

Given two variables, $x = \{x_1, x_2, x_3, ..., x_N\}$ and $y = \{y_1, y_2, y_3, ..., y_N\}$

$$\sum_{i=1}^{N}(x_i + y_i) = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i$$

### 1.2.2 Expected Value.

The Expected Value operator is linear. Recall, for a discrete variable with $m$ possible different values, $\{x_1, x_2, ..., x_m\}$, the expected value is defined as:

$$E(x) = \sum_{i=1}^{m} f(x_i) \cdot x_i$$

What's $E(x+y)$? Doesn't it depend on the probability distributions for $x$ and $y$? NO. $E(x)$ is a linear operator, so

$$E(x+y) = E(x) + E(y)$$

Since $E()$ is a linear operator, it radically simplifies many calculations in statistics.

## 1.3 Operators that you love which are not linear

Don't make the mistake of thinking everything that is good is also linear. Recall, for example:

$$V(x+y) = V(x) + V(y) + 2Cov(x,y)$$

Even so, we often try to "cheat" and make $V()$ act as if it were linear. How many times do you assume away the covariance term? Almost all the time in intermediate regression.

I mention this to remind you that, if you want to apply the property of linearity, YOU CAN ONLY DO SO WHEN YOU HAVE SOME EVIDENCE THAT THE OPERATOR REALLY IS LINEAR.

# 2 The Derivative is a linear operator

You might be asked to find the derivative of a sum of functions, such as

$$\frac{\partial}{\partial x}\left(f_1(x) + f_2(x)\right) = ?$$

The derivative is a linear operator, apply the "derivative operator" $\frac{\partial}{\partial x}$ to the individual terms, to find:

$$\frac{\partial f_1(x)}{\partial x} + \frac{\partial f_2(x)}{\partial x}$$

The beauty of this is that you can solve a series of small problems and add up the solutions, rather than solving one giant confusing problem. If

$$\frac{\partial f(x)}{\partial x} = 7x$$

and

$$\frac{\partial g(x)}{\partial x} = 3x^2$$

then

$$\frac{\partial}{\partial x}\left(f(x) + g(x)\right) = 7x + 3x^2$$

## 3 The Integral is also a linear operator.

One of the really handy rules is that the integral of a sum is the sum of the integrals.

$$\int \{f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx$$