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Curves and Slopes
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1 Relationships.

Consider some possible relationships in Figure 1.
X is an independent variable
Y is a dependent variable
The line represents the "expected" value of Y as it depends on X.
When you consider a relationship, you probably ought to consider some basic questions:

- is it linear?
- If it is not linear, how does the relationship depend on x?
- Are there "boundaries" or limits on the values of x and y?
- Is there a "global maximum"?
- Are there local minima or maxima?

2 Straight lines. Can't live with 'em, can't live without 'em.

2.1 Look at a straight line!

Consider the straight line graphed in Figure 2.

A point is an ordered pair that is represented by a dot in the plane. For example, (x1,y1) is a point. Ordinarily, we would use subscripts on the x's and y's, but my drawing software does not make that easy, so I'm not putting them in the text either. (My drawing software does not allow some other things either, but let's not go on singing that sad song.)

Comparing two points (x1,y1) and (x2,y2):

the difference between them horizontally is Dx=x2-x1.
the difference between them vertically is Dy=y2-y1.

It is pretty easy to see the slope of the line is the ratio of the two changes.

$$\frac{Dy}{Dx} = \frac{y2 - y1}{x2 - x1}$$

When you look at that figure, you see the slope of a straight line does not depend on how you select either x1 or x2.

2.2 Slope is a local concept, though.

Look at the broken line in Figure 3.

What ideas do you have about how to define or interpret slope in this case?

3 Slope of a "smooth curve"

Look at the smooth curve I drew in Figure 4.

This is drawn so "x" is a particular point and we pick various values x3,x2,x1, that get closer and closer to x. Note how the slope $\frac{Dy}{Dx}$ adjusts every time the reference point xi changes.

4 A Derivative is the result of an “itty bitty” change in X.

4.1 Keep making Dx smaller and smaller!

In Figure 5, it shows what would happen if we kept taking values of xi and making them get closer and closer to x. Eventually, the slope would approach a “limiting value”. The limiting value is the slope of a tangent line.

4.2 A “tangent line” is...

Well, I don’t actually know the technical definition. But I know a tangent line one when I see one! The tangent line “just touches” the curve at x. You create the tangent line by making Dx smaller and smaller. When Dx gets “arbitrarily small” (that means really really small, like as small as you can possibly imagine), then

4.3 Derivative.

There is a special name for the slope of that tangent line. It is called the derivative of f at x.

I don’t expect any math professors will ever read this, so lets just leave it at that.

There are various notations people use for a derivative.

One classic notation is:

$$\frac{df(x)}{dx}$$

or, if it is already known that $y = f(x)$ then:

$$\frac{dy}{dx}$$

or sometimes people don’t want to refer to y, so they say “f prime of x”, as in:

$$f'(x)$$

or if you are an “operator” minded person, think of D as the derivative operator:

$$Df(x)$$

In case you are a math professor, why don’t you take an afternoon and write a simple thing about continuity and limits and email me a reference :).

5 Derivative uses:

5.1 Describe a relationship:

Look back at Figure 1. For each one, say something like “the effect of x on y is small when x is small, but bigger when x is in the middle ranges, and smaller when x is large.”

5.2 Find local maxima and minima

The single most important use of derivatives is to check to see if f(x) is a local maximum or minimum.

See why?

Hint: What is the derivative at a maximum or minimum point?

Gotta be 0. Right?

So, if you read in any kind of statistics or economics or physics book, they spend a lot of time/effort translating something into a function that depends on or more variables, and then they find the place where they find the “critical points”, the points for which the slope is 0.

5.3 Just a little bit more: second derivatives.

The second derivative is the change in the slope if you start at x and go an “itty bitty” amount to the right.

If the slope is **getting bigger**, that means the impact of x is “accelerating”.

If the slope is **getting** smaller, that means the impact of x is “decelerating” or has “diminishing impact”.

Sometimes notations for the change in the slope are:

$$f''(x) = D^2 f(x) = \frac{d^2y}{dx^2} = \frac{d^2 f(x)}{dx^2}$$

For some intangible reason, I like the notation: f”(x).

If the slope is “getting smaller,” then

Think for a minute about each of these claims.

1. If $f'(x) = 0$ and $f''(x) < 0$, then x is a local maximum.
2. If $f'(x) = 0$ and $f''(x) > 0$, then x is a local minimum.

6 What to do if you want to learn more

If you get any elementary book about calculus, you will find a couple of chapters devoted to the development of the ideas described here. Two particularly important “foundation” ideas are *limit* and *continuous function*.

There are formulas and procedures for calculating the derivative of a function. Some are stupendously easy!

7 What to do if you don’t want to learn more.

Don’t worry. All you have to know is that

1. derivative means “slope” of a function. Slope means “effect” and many of our social theories require us to describe the effect of x on y.
2. if the derivative is 0, we are at a critical point that might be a maximum or a minimum.

Figure 1: Some Curves

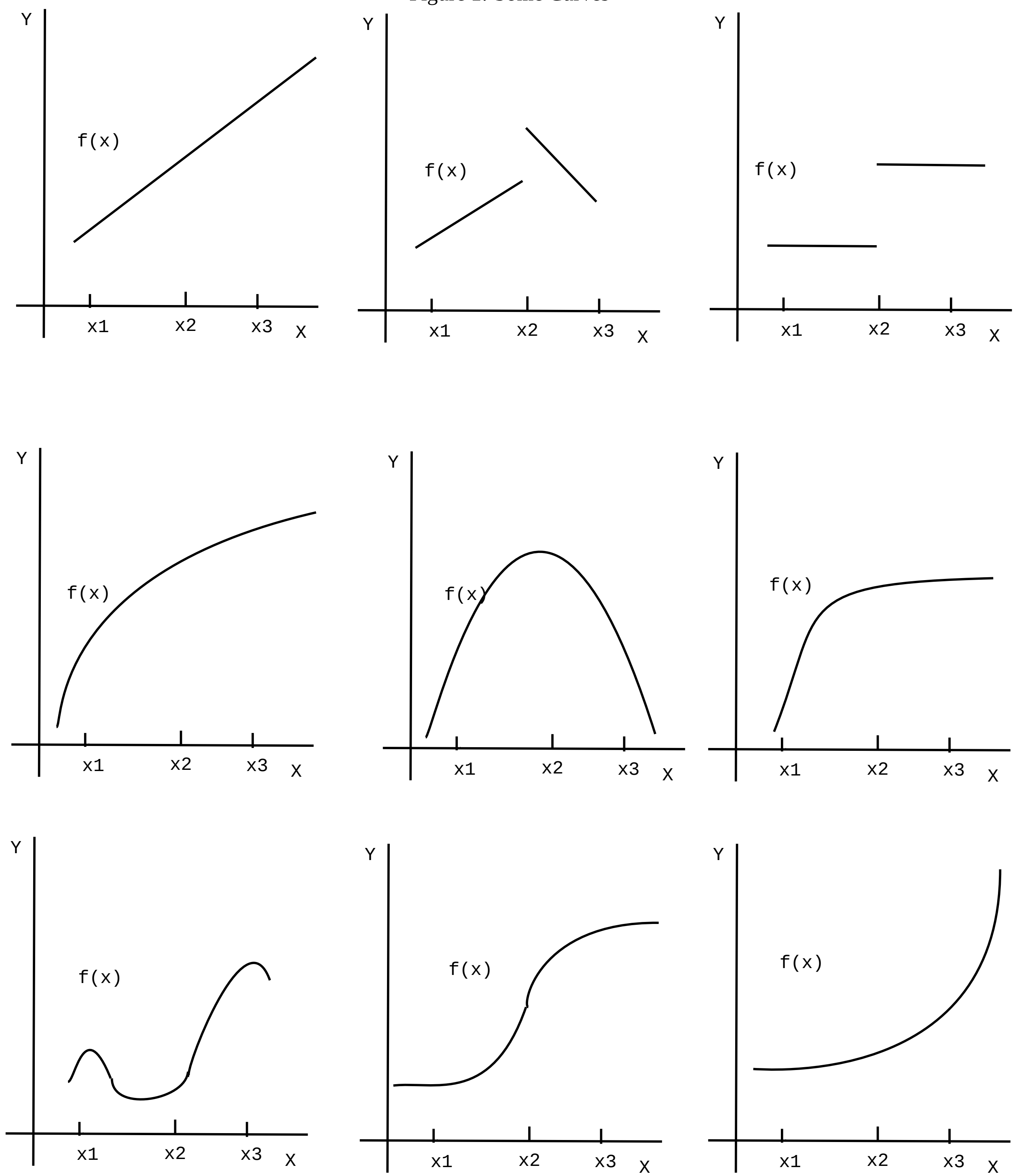

Figure 2: Slope of a straight line

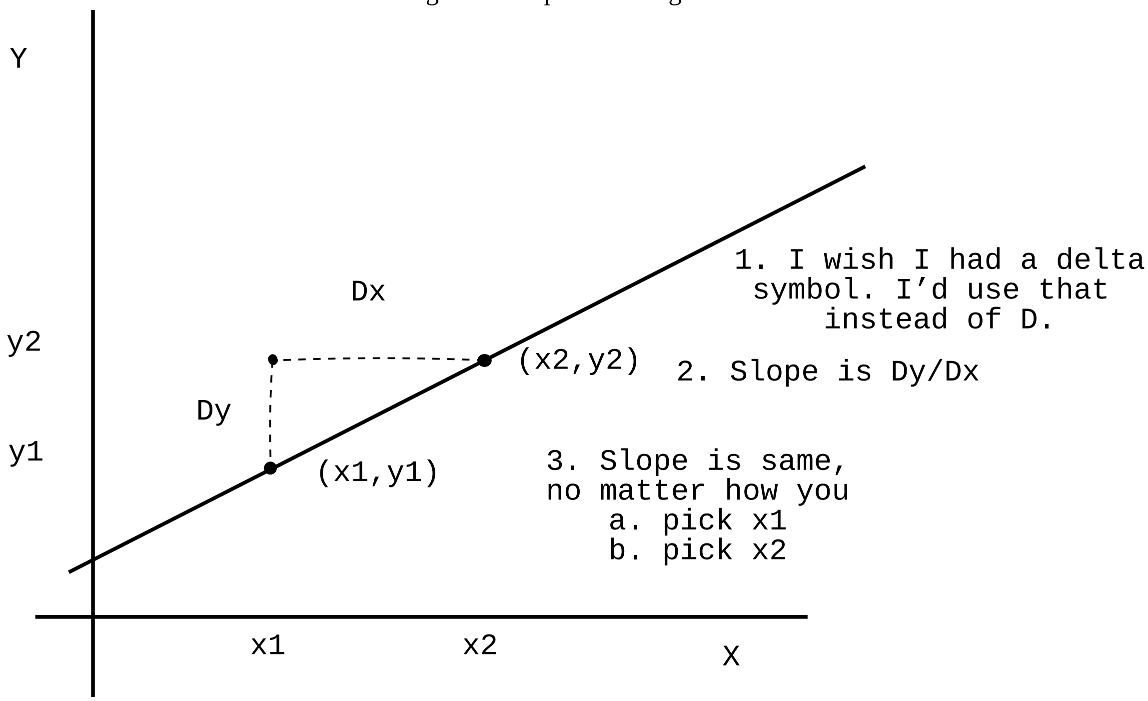

Figure 3: Slope is a Local Concept

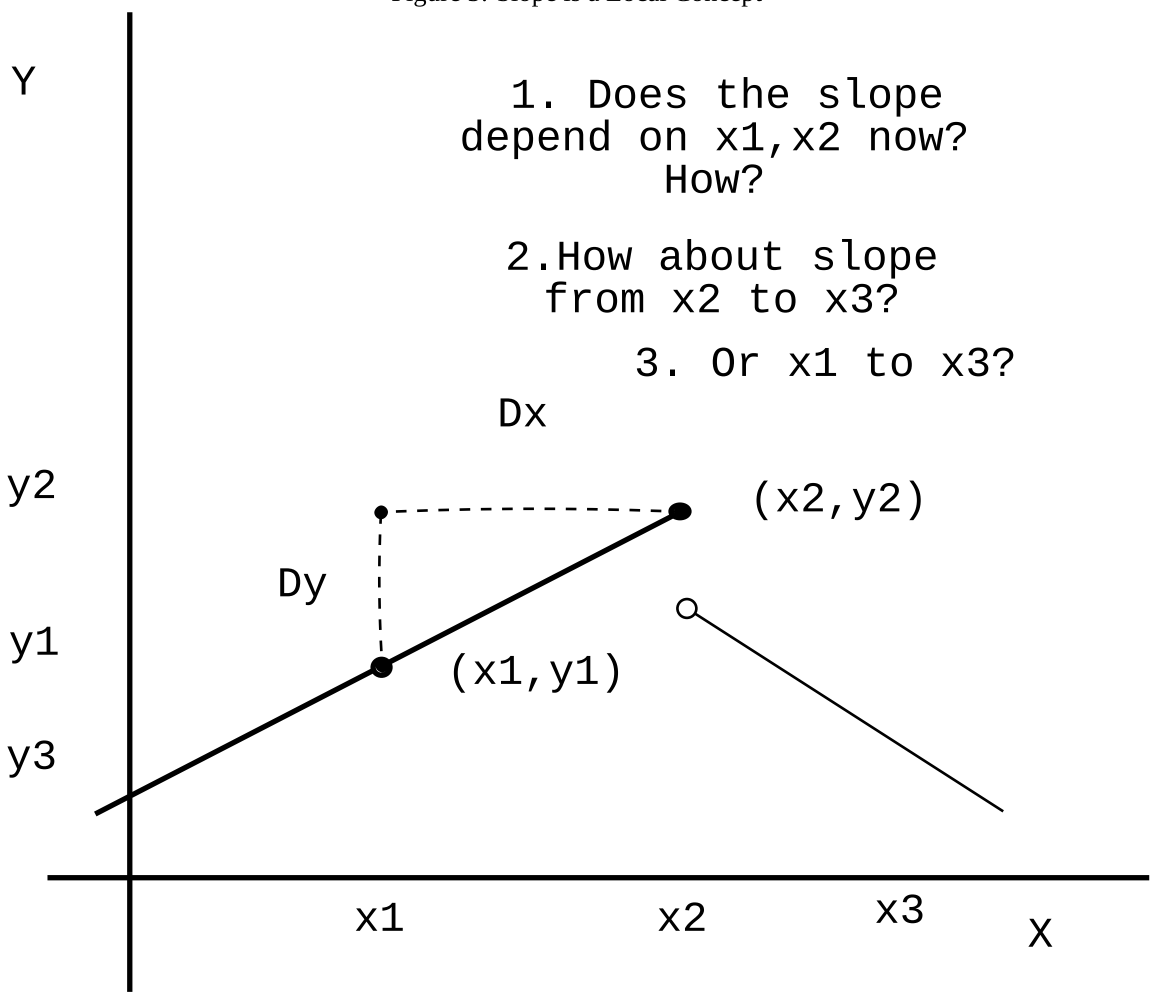

Figure 4: Make Dx smaller and smaller

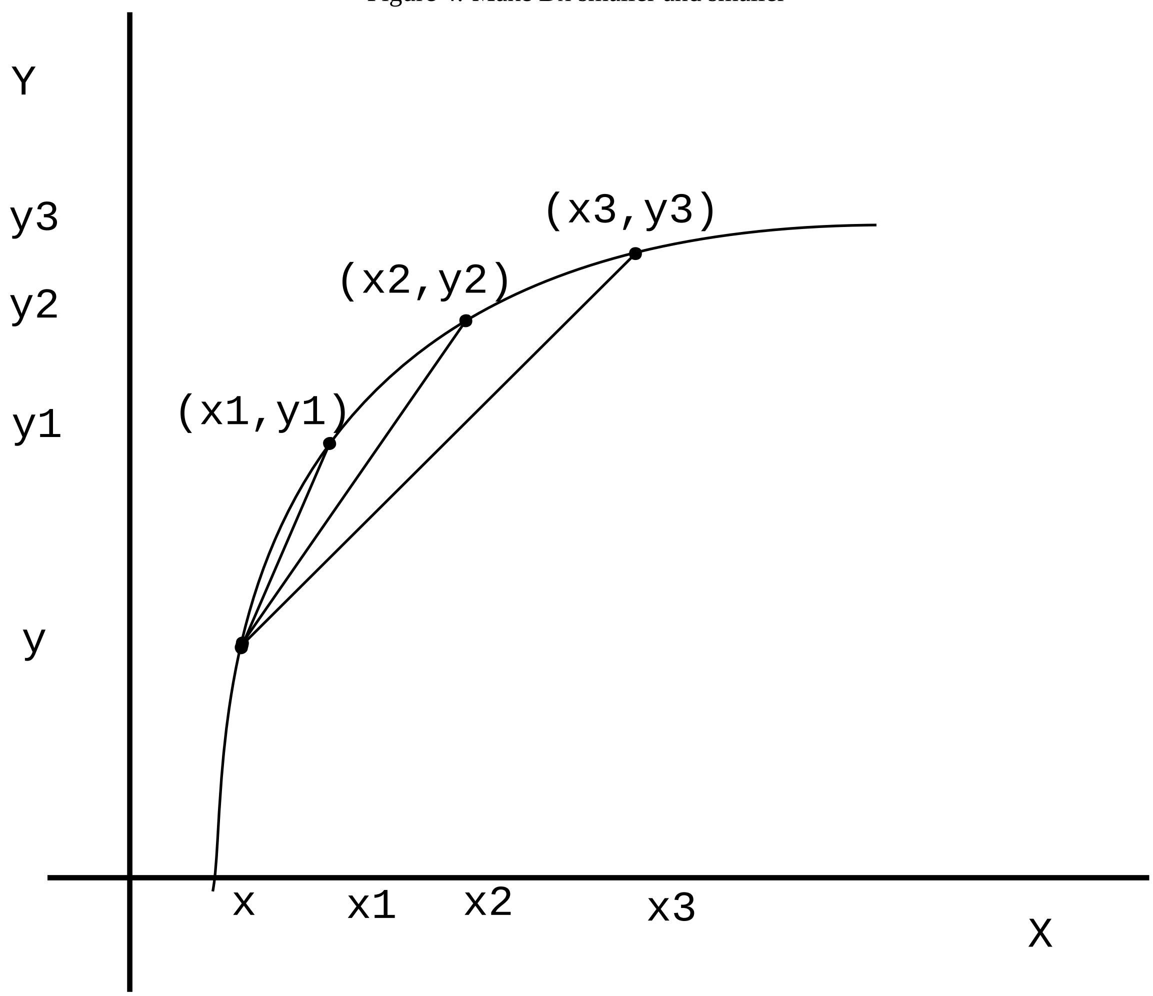

Figure 5: What if Dx gets really really small?