Psychological Bulletin
1984, Vol. 96, No. 1, 201–210

Estimating the Nonlinear and Interactive Effects of Latent Variables

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A procedure is described that enables researchers to estimate nonlinear and interactive effects of latent variables in structural equation models. Given that the latent variables are normally distributed, the parameters of such models can be estimated. To do this, products of the measured variables are used as indicators of latent product variables. Estimation must be done using a procedure that allows nonlinear constraints on parameters. The procedure is demonstrated in three different examples. The first two use artificial data with known parameter values. These parameters are successfully recovered by the procedure. The final complex example uses national election survey data.

The use of structural models with latent or unmeasured variables (Bentler, 1980; Maruyama & McGarvey, 1980) is increasing in the social sciences. Such models are useful because they allow us to estimate the coefficients of linear models while controlling for the biasing effects of measurement error. One useful but oversimplified view of latent variable structural models is that they involve two estimation procedures. First, an oblique factor analysis is performed. Second, the covariances among the resulting factors are entered into a multiple regression procedure. The estimation of latent variable structural models can be viewed as a synthesis of factor analysis and multiple regression, with both estimation procedures conducted simultaneously.

However, there are two features in estimating the coefficients of linear models with multiple regression that are not available in latent variable models. With multiple regression, it is a relatively simple matter to estimate the nonlinear and interactive effects of predictor or exogenous variables. We do this by computing the appropriate product terms among the exogenous variables and then entering those products as predictors in a regression equation. Thus, if X and Z are viewed as exogenous variables that affect Y, we estimate the nonlinear effect of X on Y with the equation:

$$Y = aX + bX^2 + W, \tag{1}$$

and the interactive effects of X and Z on Y with the equation:

$$Y = cX + dZ + eXZ + V. \tag{2}$$

In these equations a, b, c, d, and e are regression coefficients; W and V are the usual residual terms in regression equations.

Models with interactions and nonlinear effects are quite common in psychology. Busemeyer and Jones (1983) have shown that there is currently no adequate procedure available to estimate interactive and nonlinear effects of latent variables. They have also shown that the reliability of product terms tends to be less than the reliability of the component variables. Hence, a procedure to estimate nonlinear and interactive effects of latent variables would be quite useful.

The purpose of this article is to demonstrate such a procedure. That is, we show how the coefficients for Equations 1 and 2 can be estimated when X and Z, and consequently X^2 and XZ, are unmeasured or latent variables. We first explain the estimation with simple nonlinear and interactive models, using computer generated data to illustrate the techniques. We then illustrate the procedure with a more complex example using national survey

This research was supported in part by National Science Foundation Grant BNS8210127 to David A. Kenny. The authors wish to thank the Social Science Data Analysis Center at the University of Colorado and the Interuniversity Consortium for Political and Social Research for providing access to data used in this article.

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data from the 1968 National Election Survey conducted by the Center for Political Studies at the University of Michigan.

Nonlinear Effects

We begin by showing how to estimate the coefficients of Equation 1 when X is a latent variable. The measured variables, X_1 and X_2, are indicators of latent variable X. In equation form,

$$X_1 = X + U_1 \tag{3}$$

$$X_2 = fX + U_2, \tag{4}$$

with U_1, U_2, W, and X all uncorrelated, W being the residual in Equation 1. All variables are in mean deviation form. That is, the means of U_1, U_2, W, and X are zero.

To estimate the effects of the latent variable X^2, we need to develop indicators of it. We can use the three possible products among the indicators of X as indicators of X^2. Thus, X_1^2, X_2^2, and X_1X_2 are all indicators of X^2. These products can be expressed as functions of latent variables by taking the appropriate products of Equations 3 and 4:

$$X_1^2 = X^2 + 2XU_1 + U_1^2, \tag{5}$$

$$X_2^2 = f^2X^2 + 2fXU_2 + U_2^2, \tag{6}$$

$$X_1X_2 = fX^2 + fXU_1 + XU_2 + U_1U_2. \tag{7}$$

Equations 3 through 7 imply the loading matrix contained in Table 1. This matrix contains the loadings of the indicators on all latent variables. As can be seen, there is only one free parameter or loading coefficient to be estimated, that is, f, the loading of X_2 on X. All other nonzero loadings are either set at one or two or are functions of f. Thus, when the nonlinear indicators (X_1^2, X_2^2, and X_1X_2) are included, no new loading coefficients need to be estimated.

The loadings of the nonlinear indicators are derived by simple algebraic manipulations performed on Equations 3 and 4 without involving any additional distributional assumptions. The covariance matrix among the latent variables X, X^2, U_1, U_2, U_1^2, U_2^2, XU_1, XU_2, and U_1U_2, however, can be known only if we make further distributional assumptions. That is, different distributions of the latent variables, X, U_1, U_2, result in different covariance matrices. Following Bohrnstedt and Goldberger (1969) and Busemeyer and Jones (1983), we assume that the latent variables X, U_1, and U_2 are normally distributed. Under this assumption, it follows that

$$\sigma^2_{X^2} = 2\sigma^4_X; \quad \sigma^2_{U_1^2} = 2\sigma^4_{U_1};$$

$$\sigma^2_{U_2^2} = 2\sigma^4_{U_2}; \quad \sigma^2_{XU_1} = \sigma^2_X\sigma^2_{U_1};$$

$$\sigma^2_{XU_2} = \sigma^2_X\sigma^2_{U_2}; \quad \sigma^2_{U_1U_2} = \sigma^2_{U_1}\sigma^2_{U_2}.$$

It also follows that all covariances between the latent variables[1] are zero (see Appendix). Given the normality assumption, with the previous assumptions that X, U_1, and U_2 are all uncorrelated with zero means, then the variances of all of the other latent variables are functions of σ^2_X, $\sigma^2_{U_1}$, and $\sigma^2_{U_2}$. Thus, the model with nonlinear indicators of X^2 is in principle identified, because no new parameters need to be estimated outside of the effect on X^2 on Y.

Although we have assumed that X, U_1, and U_2 are normally distributed, latent variables that are products of these (e.g., X^2) cannot be normally distributed (Kendall & Stuart, 1958). A frequent assumption in estimating latent variable models is that all variables are normally distributed. Here this assumption clearly does not hold. Hence, we cannot use a procedure that estimates parameters by minimizing a maximum likelihood loss function that assumes multivariate normality of the latent variables. For example, the maximum likelihood estimation procedure of LISREL (Jöreskog & Sörbom, 1981) is inappropriate. McDonald (1978) suggests that a reasonable alternative is a generalized least squares loss function. Therefore, we use a generalized least squares estimation procedure in which the weighting matrix is the inverse of the sample covariance matrix (Fraser, 1980).

There is one further complication in the estimation. Nonlinear constraints must be put on the estimated parameters. For instance, both f and f^2 must be estimated. Likewise, variances that are products of σ^2_X, $\sigma^2_{U_1}$, and

[1] There is, throughout this article, the potential to confuse the variance of a product variable, for example, σ^2_{XY}, and the squared covariance of its components, for example, $\sigma^2_{X,Y}$. As shown, we differentiate between these two by using a comma between the component variables involved in covariances.

Table 1
Loading Matrix for Nonlinear Model

Variable	X	X^2	U_1	U_2	U_1^2	U_2^2	XU_1	XU_2	U_1U_2
X_1	1	0	1	0	0	0	0	0	0
X_2	f	0	0	1	0	0	0	0	0
X_1^2	0	1	0	0	1	0	2	0	0
X_2^2	0	f^2	0	0	0	1	0	$2f$	0
X_1X_2	0	f	0	0	0	0	f	1	1

$\sigma^2_{U_2}$ must be estimated. McDonald (1978) describes a procedure for estimating latent variable models that allows nonlinear constraints to be placed on the coefficients. Fraser (1980) has written the program (COSAN) that implements McDonald's ideas.

Example

To illustrate the estimation, we used a random number generator to create values for 500 cases on the four latent variables, X, U_1, U_2, and W. These variables were generated so as to be uncorrelated, multivariate normal, with zero means and variances of 1.0, 0.15, 0.55, and 0.20, respectively, in the population. From these four variables, we derived values for the 500 cases for X_1, X_2, X_1^2, X_2^2, X_1X_2, and Y. In generating these values, the following coefficients were used: $a = .25$, $b = -.50$, and $f = .60$. Thus, the data were derived from the following set of population equations:

$$Y = .25X - .50X^2 + W,$$

$$X_1 = X + U_1,$$

and

$$X_2 = .60X + U_2.$$

Our task is to show that these coefficients can be recovered from the estimation procedure performed on the observed sample covariance matrix among X_1, X_2, X_1^2, X_2^2, X_1X_2, and Y. We assume that Y is perfectly measured. This assumption does not, however, limit the generality of the procedure. As we illustrate later in the example with real data, Y can also be a latent variable.

The resulting sample covariance matrix among the observed variables is presented in Table 2.

Using COSAN, the following generalized least squares estimates of the parameters were obtained: $a = 0.247$ $b = -0.500$ $f = 0.624$ $\sigma^2_X = 0.989$ $\sigma^2_{U_1} = 0.160$ $\sigma^2_{U_2} = 0.540$ $\sigma^2_W = 0.199$. It seems to us that the procedure recovered the coefficients quite accurately.

Interactive Effects

Interactions among latent variables are handled similarly to nonlinear effects. Indicators of the interactions are formed, and their loading matrix is derived by simple algebra. The covariance matrix among the latent variables is derived under the assumption of multivariate normality.

The interactive model of Equation 2 has Y affected by X, Z, and XZ. The latent variables X and Z have two indicators each. Their equations are

$$X_1 = X + U_1, \tag{8}$$

$$X_2 = gX + U_2, \tag{9}$$

$$Z_1 = Z + U_3, \tag{10}$$

$$Z_2 = hZ + U_4. \tag{11}$$

The indicators of the XZ product latent variable are

$$X_1Z_1 = XZ + XU_3 + ZU_1 + U_1U_3, \tag{12}$$

$$X_1Z_2 = hXZ + XU_4 + hZU_1 + U_1U_4, \tag{13}$$

$$X_2Z_1 = gXZ + gXU_3 + ZU_2 + U_2U_3, \tag{14}$$

$$X_2Z_2 = ghXZ + gXU_4 + hZU_2 + U_2U_4. \tag{15}$$

Thus, there are a total of 15 latent variables: X, Z, XZ, XU_3, XU_4, ZU_1, ZU_2, U_1, U_2, U_3, U_4, U_1U_3, U_1U_4, U_2U_3, and U_2U_4. The loading matrix for the 8 observed variables on these 15 latent variables is contained in Table 3. This matrix appears complex; however, there are in fact only 2 free parameters to be estimated: g and h.

Table 2
*Nonlinear Example: Observed Sample Covariance Matrix (*N *= 500)*

Variable	X_1	X_2	X_1^2	X_2^2	X_1X_2	Y
X_1	1.150					
X_2	0.617	0.981				
X_1^2	−0.068	−0.025	2.708			
X_2^2	0.075	0.159	0.729	1.717		
X_1X_2	0.063	0.065	1.459	1.142	1.484	
Y	0.256	0.166	−1.017	−0.340	−0.610	0.763

Assuming once again that X, Z, U_1, U_2, U_3, U_4, and V are all in mean deviation form, multivariate normal, and mutually uncorrelated with the exception of X and Z, the diagonal of the covariance matrix among the latent variables is

$$\sigma^2_{XZ} = \sigma^2_X\sigma^2_Z + \sigma^2_{X,Z}$$

$$\sigma^2_{U_1U_3} = \sigma^2_{U_1}\sigma^2_{U_3}$$

$$\sigma^2_{U_1U_4} = \sigma^2_{U_1}\sigma^2_{U_4}$$

$$\sigma^2_{U_2U_3} = \sigma^2_{U_2}\sigma^2_{U_3}$$

$$\sigma^2_{U_2U_4} = \sigma^2_{U_2}\sigma^2_{U_4}$$

$$\sigma^2_{XU_3} = \sigma^2_X\sigma^2_{U_3}$$

$$\sigma^2_{XU_4} = \sigma^2_X\sigma^2_{U_4}$$

$$\sigma^2_{ZU_1} = \sigma^2_Z\sigma^2_{U_1}$$

$$\sigma^2_{ZU_2} = \sigma^2_Z\sigma^2_{U_2},$$

where $\sigma_{X,Z}$ is the covariance of X and Z (see Appendix). The only nonzero covariance in the matrix is $\sigma_{X,Z}$. Again, the program COSAN, using a generalized least squares loss function, can be used to estimate the coefficients of the model under these nonlinear constraints.

Example

To illustrate the estimation, we once again generated values for 500 cases on the 7 latent variables X, Z, U_1, U_2, U_3, U_4, and V. All variables were generated so that in the population they had means of zero and shared a multivariate normal distribution. All pairs of variables were uncorrelated in the population with the exception of X and Z, which were generated so that their correlation in the population was .20. The population variances for the 7 variables were

$$\sigma^2_X = 2.15$$

$$\sigma^2_{U_1} = 0.36$$

$$\sigma^2_{U_3} = 0.49$$

$$\sigma^2_V = 0.16$$

$$\sigma^2_Z = 1.60$$

$$\sigma^2_{U_2} = 0.81$$

$$\sigma^2_{U_4} = 0.64.$$

From these latent variables, we derived values for the five hundred cases for X_1, X_2, Z_1, Z_2, X_1Z_1, X_1Z_2, X_2Z_1, X_2Z_2, and Y. In gen-

Table 3
Loading Matrix for Interactive Model

Variable	X	Z	XZ	U_1	U_2	U_3	U_4	U_1U_3	U_1U_4	U_2U_3	U_2U_4	XU_3	XU_4	ZU_1	ZU_2
X_1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
X_2	g	0	0	0	1	0	0	0	0	0	0	0	0	0	0
Z_1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
Z_2	0	h	0	0	0	0	1	0	0	0	0	0	0	0	0
X_1Z_1	0	0	1	0	0	0	0	1	0	0	0	1	0	1	0
X_1Z_2	0	0	h	0	0	0	0	0	1	0	0	0	1	h	0
X_2Z_1	0	0	g	0	0	0	0	0	0	1	0	g	0	0	1
X_2Z_2	0	0	gh	0	0	0	0	0	0	0	1	0	g	0	h

Table 4
Interactive Example: Observed Sample Covariance Matrix (N = 500)

Variable	X_1	X_2	Z_1	Z_2	X_1Z_1	X_1Z_2	X_2Z_1	X_2Z_2	Y
X_1	2.395								
X_2	1.254	1.542							
Z_1	0.445	0.202	2.097						
Z_2	0.231	0.116	1.141	1.370					
X_1Z_1	−0.367	−0.070	−0.148	−0.133	5.669				
X_1Z_2	−0.301	−0.041	−0.130	−0.117	2.868	3.076			
X_2Z_1	−0.081	−0.054	0.038	0.037	2.989	1.346	3.411		
X_2Z_2	−0.047	−0.045	0.039	−0.043	1.341	1.392	1.719	1.960	
Y	−0.368	−0.179	0.402	0.282	2.556	1.579	1.623	0.971	2.174

erating these nine observed variables, the following coefficients were used: $c = -.15$, $d = .35$, $e = .70$, $g = .60$, and $h = .70$. Thus, the data were derived from the following set of population equations:

$$Y = -.15X + .35Z + .70XZ + V,$$

$$X_1 = X + U_1,$$

$$X_2 = .60X + U_2,$$

$$Z_1 = Z + U_3,$$

$$Z_2 = .70Z + U_4.$$

Once again, our task is to show that these coefficients can be recovered within the limits of sampling error from the estimation procedure performed on the observed sample covariance matrix. That matrix is contained in Table 4.

Using COSAN, we obtained the following generalized least squares estimates of the free parameters:

$$c = -0.169 \quad d = 0.321 \quad e = 0.710$$
$$g = 0.646 \quad h = 0.685$$
$$\sigma_X^2 = 1.883 \quad \sigma_Z^2 = 1.654 \quad \sigma_{X,Z} = 0.369$$
$$\sigma_{U_1}^2 = 0.428 \quad \sigma_{U_2}^2 = 0.721 \quad \sigma_{U_3}^2 = 0.444$$
$$\sigma_{U_4}^2 = 0.552 \quad \sigma_V^2 = 0.265$$

Once again, the procedure seems to have recovered the generating coefficients.

Complex Example

We now study an example using real data. The reader should be forewarned that we are using a complex example to illustrate the full potential of the procedure. A number of researchers in social psychology have recently been interested in the extent to which voters misperceive the positions espoused by political candidates (e.g., Granberg & Brent, 1974; Granberg & Seidel, 1976; Judd, Kenny, & Krosnick, 1983; Kinder, 1978). More specifically, they have examined whether voters assimilate and contrast the positions of candidates whom they either like or dislike. Assimilation would be found if voters overestimate their agreement with liked candidates. Contrast would be found if voters overestimate their disagreement with disliked candidates. Both assimilation and contrast are consistent with balance theory.

The hypothesis of assimilation and contrast argues that the relation between a voter's position on an issue (V) and his or her judgment of the candidate's position (C) should be moderated by the voter's liking or sentiment (S) toward the candidate. If the candidate is disliked, a negative relation between V and C is consistent with contrast. If the candidate is liked, a positive V–C relation is consistent with assimilation. Hence, the voter's own position (V) and his or her sentiment toward the candidate (S) should interact to affect the judgment of the candidate's position (C).

Some of the early work on the assimilation-contrast hypothesis suggested that assimilation effects are more potent than contrast effects. This suggestion means that the effect of the VS interaction on C should be stronger at higher levels of S. In other words, not only should the VS interaction affect C, but also the VS^2 interaction should affect C.

Judd et al. (1983) argued that misspecifications in the existing research were probably responsible for the conclusion that VS^2 affects C. Some of these misspecifications, such as

Table 5
Loading Matrix for Assimilation-Contrast Example

Variable	V	S	S^2	VS	VS^2	U_1	U_2	U_1S	U_2S	U_1S^2	U_2S^2
V_1	1	0	0	0	0	1	0	0	0	0	0
V_2	1	0	0	0	0	0	1	0	0	0	0
S	0	1	0	0	0	0	0	0	0	0	0
S^2	0	0	1	0	0	0	0	0	0	0	0
V_1S	0	0	0	1	0	0	0	1	0	0	0
V_2S	0	0	0	1	0	0	0	0	1	0	0
V_1S^2	0	0	0	0	1	0	0	0	0	1	0
V_2S^2	0	0	0	0	1	0	0	0	0	0	1

the probable presence of correlated measurement errors in V and C, can be eliminated if V and C are treated as latent variables, with multiple indicators of each, allowing errors of measurement in indicators of V to covary with errors of measurement in indicators of C.

Using the 1968 National Election Survey conducted by the Center for Political Studies of the University of Michigan, Judd et al. (1983) examined the judgment of the presidential candidates Hubert Humphrey and Richard Nixon on two issues: the Vietnam War and control of crime. Using a latent variable model, Judd et al. examined the VS and VS^2 interactions by dividing up the sample on sentiment (S) toward the candidate and looking for both linear and nonlinear differences in the path from V to C among the sentiment subsamples. Using this procedure, Judd et al. found strong evidence for the VS interaction but no support for the effect of VS^2. In other words, assimilation-contrast was found, but no evidence was found for stronger assimilation than contrast.

A much more efficient procedure to examine these issues is to estimate the effects of the VS and VS^2 interactions directly in the latent variable model. In the model, there are two indicators of V and two of C for each candidate. V_1 and C_1 are voters' judgments of self and candidate on the crime issue. V_2 and C_2 are judgments on the Vietnam War issue. They are assumed to be indicators of latent constructs V and C. These are defined as the underlying ideological position of the voters and the judged ideological position of the candidates. Errors of measurement in V_1 and V_2 are allowed to affect errors in C_1 and C_2, respectively. Sentiment (S) was measured directly on a 100-point "thermometer" scale. (See Judd et al., 1983, for a thorough definition of all variables.)

The model's equations are as follows: First, V_1 and V_2 are indicators of V, and C_1 and C_2 are indicators of C:

$$V_1 = V + U_1,$$

$$V_2 = V + U_2,$$

$$C_1 = C + U_3, \quad \text{and}$$

$$C_2 = C + U_4.$$

Notice that all loading coefficients here are set at one. This constraint is necessary for the model to be identified when there are only two indicators each of V and C and when their errors are allowed to correlate.[2] The constraint is also justified by earlier research that has shown that attitudes on various political issues have approximately equal loadings on a single underlying construct (Judd & Milburn, 1980). The structural equation among these latent variables is

$$C = aV + bS + cS^2 + dVS + eVS^2 + W, \quad (16)$$

where a through e are parameters to be estimated and W is the usual disturbance term,

[2] The necessity of this constraint to achieve the model's identification is not a result of the nonlinear estimation that we are conducting. The constraint is necessary with only two indicators of V and C regardless of whether nonlinear terms are present. There is nothing in our procedures that constrains the loadings of observed variables over and above any constraints necessary for a model to be identified in the absence of nonlinear effects.

Table 6
Paths for the Assimilation-Contrast Example

Latent endogenous variables	Latent exogenous variables										
	V	S	S^2	VS	VS^2	U_1	U_2	U_1S	U_2S	U_1S^2	U_2S^2
C	X	X	X	X	X	0	0	0	0	0	0
U_3	0	0	0	0	0	X	0	X	0	X	0
U_4	0	0	0	0	0	0	X	0	X	0	X

assumed to be uncorrelated with all exogenous variables.

Table 5 presents the loading matrix for the indicators of the latent exogenous variables. Notice that once the loading of V_1 and V_2 on V are constrained at one, all other loadings in this matrix are also constrained. Table 6 presents the paths from the latent exogenous constructs to the latent endogenous constructs that need to be estimated. All X entries in this table are free parameters to be estimated. The five Xs in the first row of the table represent the parameters a to e in Equation 16. The other free parameters allow for correlated errors of measurement between U_1 and U_3 and between U_2 and U_4. Notice that we are allowing the magnitude of the relation between errors to vary both linearly and nonlinearly with S.

Table 7 presents the nonzero parameters in the covariance matrix of the latent exogenous constructs. These terms were generated under the assumption that all variables are normally distributed with means of zero. (See Appendix.) In fact, however, we know that the normality assumption is not true. Sentiment (S), for instance, as a directly measured variable is nonnormal. Consequently, the constraints on the covariance matrix in Table 7 are in error to the extent that the distributions are not normal. Nevertheless, it is instructive to illustrate the procedure with these data even though we know its assumptions are violated.

The observed variables S, V_1, V_2, C_1, and C_2 are all in mean deviation form. The product indicators are not. In addition, sentiment scores are divided by 10 so that the latent variables involving S^2 will have more manageable variances and covariances.

Table 8 contains the covariance matrix for the 10 observed variables for the Nixon model. The sample size is 1,160, consisting of all respondents to the 1968 election study who provided complete data on all relevant variables.

COSAN was used to provide generalized least squares estimates for all parameters. The estimated paths from the latent exogenous variables to the latent endogenous variables are presented in Table 9. The estimates of the free variances and covariances of the latent variables are

$$\sigma_V^2 = 0.894 \quad \sigma_S^2 = 4.208 \quad \sigma_{V,S} = 0.095$$
$$\sigma_{U_1}^2 = 2.456 \quad \sigma_{U_2}^2 = 2.629 \quad \sigma_{U_3}^2 = 1.619$$
$$\sigma_{U_4}^2 = 1.363 \quad \sigma_W^2 = 3.74$$

Returning to the parameter estimates in Table 9, it can be seen that the estimated coef-

Table 7
Variances and Covariances of Latent Variables for Assimilation-Contrast Example

$$\sigma_{S^2}^2 = 2\sigma_S^4$$
$$\sigma_{VS}^2 = \sigma_V^2\sigma_S^2 + \sigma_{V,S}^2$$
$$\sigma_{VS^2}^2 = 3\sigma_V^2\sigma_S^4 + 12\sigma_{V,S}^2\sigma_S^2$$
$$\sigma_{V,VS^2} = \sigma_V^2\sigma_S^2 + 2\sigma_{V,S}^2$$
$$\sigma_{VS,S^2} = 2\sigma_{V,S}\sigma_S^2$$
$$\sigma_{S,VS^2} = 3\sigma_{V,S}\sigma_S^2$$
$$\sigma_{U_1S}^2 = \sigma_{U_1}^2\sigma_S^2$$
$$\sigma_{U_2S}^2 = \sigma_{U_2}^2\sigma_S^2$$
$$\sigma_{U_1S^2}^2 = 3\sigma_{U_1}^2\sigma_S^4$$
$$\sigma_{U_2S^2}^2 = 3\sigma_{U_2}^2\sigma_S^4$$
$$\sigma_{U_1,U_1S^2} = \sigma_{U_1}^2\sigma_S^2$$
$$\sigma_{U_2,U_2S^2} = \sigma_{U_2}^2\sigma_S^2$$

Table 8
Nixon Model: Observed Sample Covariance Matrix (N = *1,160)*

Variable	C_1	C_2	V_1	V_2	S	S^2	V_1S	V_2S	V_1S^2	V_2S^2
C_1	2.626									
C_2	0.569	2.207								
V_1	0.615	0.211	3.729							
V_2	−0.105	0.721	1.084	3.834						
S	−0.724	−0.054	0.416	0.358	4.963					
S^2	0.252	−0.219	−0.629	−0.182	−7.315	53.924				
V_1S	2.317	0.686	−0.677	−1.242	−0.629	4.071	21.080			
V_2S	1.321	2.313	−1.242	−0.685	−0.183	1.507	7.577	23.545		
V_1S^2	0.559	−0.097	21.253	7.726	6.135	−25.205	−41.522	−28.734	381.402	
V_2S^2	−4.015	1.076	7.725	23.673	3.284	−6.781	−28.884	−49.602	181.204	460.865

ficient for the VS^2 interaction is quite small. If we graph the effect of V on C at varying levels of S, that graph is exceedingly linear and quite consistent with the results presented in Judd et al.'s (1983) Figure 4.

Discussion

The purpose of this article was to suggest a way in which nonlinear and interactive effects of latent variables can be estimated. Estimating these effects is made possible by using nonlinear and product indicators. The loadings of these indicators on the latent variables are derived by multiplying together structural equations. This results in no additional loadings to be estimated. In order to derive the covariance matrix among the latent variables, distributional assumptions must be made about the latent variables.[3] These assumptions permit us to derive the variances and covariances of the latent variables that are products of other latent variables. We have assumed that the nonproduct latent variables are normally distributed with zero expected values. This is one of a set of possible assumptions that we could have made in order to derive the covariance matrix among the latent variables. In the Appendix we have shown how this assumption permits us to know the product variances and covariances. Although the normality assumption may be reasonable in some situations, in others it may be less so. For instance, in our third example we knew that the distribution of S was far from normal. In such cases, other distributional assumptions might be made to derive the product variances and covariances. If our recommended procedure is to be useful when dealing with nonnormally distributed data, derivations for other distributions need to be developed.

Although we have assumed that nonproduct latent variables are normally distributed, this assumption means that the product latent variables are not. This fact means that in estimation, we should avoid minimizing a loss function that assumes multivariate normality, such as the maximum likelihood function in LISREL (Jöreskog & Sörbom, 1981). We have therefore reported results based on a generalized least squares loss function. It is interesting to note, however, that when we estimated the parameters using a maximum likelihood criterion, the parameter estimates were in most cases not appreciably different from the generalized least squares estimates that we report. Investigations of when different loss functions result in appreciably different parameter estimates are called for.

Using a generalized least squares loss function, as opposed to a maximum likelihood one, means that the standard errors of the estimated coefficients are unknown. Thus, at this point, whereas our procedure can be used to estimate nonlinear and interaction coefficients, confidence intervals for the population values of these coefficients cannot be estimated. Therefore, we suggest that estimation of these effects should proceed only when there is clear prior

[3] It might seem that a possible distribution-free procedure would be to estimate the variances and covariances of the latent variables rather than constraining them at particular values specified by the distributional assumptions. Such an approach, however, invariably results in an unidentified model.

Table 9
Nixon Model: Estimated Effects of Exogenous Latent Variables on Endogenous Latent Variables

Endogenous latent variables	Exogenous latent variables										
	V	S	S^2	VS	VS^2	U_1	U_2	U_1S	U_2S	U_1S^2	U_2S^2
C	0.180	−0.111	−0.019	0.207	0.009	0	0	0	0	0	0
U_3	0	0	0	0	0	0.162	0	0.095	0	0.011	0
U_4	0	0	0	0	0	0	0.167	0	0.092	0	0.004

evidence for them. In the absence of known standard errors, we recommend reporting descriptive indices of a model's efficiency in reproducing a sample covariance matrix (e.g., Bentler & Bonett, 1980).

Like the normality assumption, the assumption that all nonproduct latent variables have means of zero is one possible assumption that could have been made. We could have assumed means different from zero, but this would have made the derivation of the latent product variances and covariances more complicated. When dealing with the observed variables, we have rescaled them so that their means are zero before computing the product indicators.[4] These product indicators were not, however, rescaled to have zero means. Again, we could have allowed nonproduct indicators to have nonzero means, but this raises additional complications that have yet to be fully worked out.

When faced with the need to make comparisons of nonlinear or interactive parameters across populations or over time, it is not appropriate to force the latent variables to have zero and, hence, equal means. Not only is it unlikely that the means in the different populations would be equal, but forcing equal means can greatly complicate the comparison of parameter estimates.

Our hope is that the procedure we have outlined will be useful to researchers who wish to estimate nonlinear and interactive effects in the presence of measurement error. We believe, however, that our procedure is merely a beginning in developing a general approach to such estimation. Further work needs to be devoted to the question of how various distributional assumptions can be used to derive the covariance matrix among latent variables. In addition, work that examines the consequences of violating those assumptions is also called for. We believe our recommended procedure constitutes a partial solution to this important problem.

[4] Actually, in the first two examples we did not subtract the sample means from the indicator variables prior to computing the product indicators, because these variables were constructed in such a way that their expected values were zero.

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Appendix

We show here the various expectations of product variables given that the component variables have a multivariate normal distribution. All of these results are well known in the statistical literature but are relatively unfamiliar to psychologists.

Let variables X, Y, Z, W, U, and V have a multivariate normal distribution with mean zero. The covariance between two variables will be denoted as σ_{XY} and the variance as σ_X^2. All odd moments, for example, $E(XYZ)$, are zero (Kendall & Stuart, 1958). The fourth moment is

$$E(XYZW) = \sigma_{XY}\sigma_{ZW} + \sigma_{XZ}\sigma_{YW} + \sigma_{XW}\sigma_{YZ} \quad \text{(A1)}$$

(Kendall & Stuart, 1958), where E is the expectation operator. It then follows that $E[\text{Cov}(XY, Z)] = E(XYZ) - E(XY)E(Z) = 0$. Therefore, $E[\text{Cov}(X^2, Z)] = 0$.

The expectation of Cov(XY, ZW) equals

$$E(XYZW) - E(XY)E(ZW),$$

which given Equation A1 equals $\sigma_{XY}\sigma_{ZW} + \sigma_{XZ}\sigma_{YW} + \sigma_{XW}\sigma_{YZ} - \sigma_{XY}\sigma_{ZW}$ or more simply $\sigma_{XZ}\sigma_{YW} + \sigma_{XW}\sigma_{YZ}$. Using this result we can show that

$$E[\text{Var}(XY)] = \sigma_X^2\sigma_Y^2 + \sigma_{X,Y}^2,$$

$$E[\text{Var}(X^2)] = 2\sigma_X^4,$$

$$E[\text{Cov}(XY, XW)] = \sigma_X^2\sigma_{YW} + \sigma_{XW}\sigma_{YX},$$

$$E[\text{Cov}(X^2, ZW)] = 2\sigma_{XZ}\sigma_{XW},$$

$$E[\text{Cov}(X^2, XW)] = 2\sigma_X^2\sigma_{XW}.$$

The expectation of Cov(X, YZW) equals

$$E(XYZW) - E(X)E(YZW),$$

which given Equation A1 equals: $\sigma_{XY}\sigma_{ZW} + \sigma_{XZ}\sigma_{YW} + \sigma_{XW}\sigma_{YZ}$. Using this result it follows that

$$E[\text{Cov}(X, XZW)] = \sigma_X^2\sigma_{ZW} + 2\sigma_{XW}\sigma_{XZ},$$

$$E[\text{Cov}(X, Y^2W)] = \sigma_Y^2\sigma_{XW} + 2\sigma_{YW}\sigma_{YX},$$

$$E[\text{Cov}(X, X^2W)] = 3\sigma_X^2\sigma_{XW},$$

$$E[\text{Cov}(X, X^3)] = 3\sigma_X^4.$$

All covariances involving five variables, for example, Cov(XY, ZWU), equal zero.

The sixth moment is:

$$E(XYZWUV) = \sigma_{XY}E(ZWUV) + \sigma_{XZ}E(YWUV) + \sigma_{XW}E(YZUV) + \sigma_{XU}E(YZWV) + \sigma_{XV}E(YZWU)$$

(Kendall & Stuart, 1958). It then can be shown that

$$E[\text{Var}(XYZ)] = \sigma_X^2\sigma_Y^2\sigma_Z^2 + 2\sigma_X^2\sigma_{Y,Z}^2 + 2\sigma_Y^2\sigma_{X,Z}^2 + 2\sigma_Z^2\sigma_{X,Y}^2 + 8\sigma_{XY}\sigma_{ZX}\sigma_{YZ},$$

and $E[\text{Var}(X^2Y)] = 3\sigma_X^4\sigma_Y^2 + 12\sigma_X^2\sigma_{X,Y}^2$.

Received September 6, 1983
Revision received February 3, 1984. ■