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Reviewed work(s):
Source: *Journal of the American Statistical Association*, Vol. 64, No. 328 (Dec., 1969), pp. 1439-1442
Published by: American Statistical Association
Stable URL: http://www.jstor.org/stable/2286081
Accessed: 07/01/2013 05:42

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ON THE EXACT COVARIANCE OF PRODUCTS OF RANDOM VARIABLES*

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For the general case of jointly distributed random variables x and y, Goodman [3] derives the exact variance of the product xy. For the special case where x and y are stochastically independent, he provides a simpler expression for the exact variance. We offer a weaker set of assumptions which suffices to yield the simpler expression. We then extend Goodman's analysis to present the exact covariance of two products xy and uv, and sketch several specializations and applications.

1. THE VARIANCE OF A PRODUCT

Let x and y be jointly distributed random variables with expectations $E(x)$ and $E(y)$, variances $V(x)$ and $V(y)$, and covariance $C(x, y)$. Consider the product xy; by definition its variance is

$$V(xy) = E[xy - E(xy)]^2. \tag{1}$$

Let $\Delta x = x - E(x)$ and $\Delta y = y - E(y)$, and write

$$\begin{aligned} xy &= [\Delta x + E(x)][\Delta y + E(y)] \\ &= (\Delta x)(\Delta y) + (\Delta x)E(y) + (\Delta y)E(x) + E(x)E(y). \end{aligned} \tag{2}$$

Take expectations to give

$$E(xy) = E[(\Delta x)(\Delta y)] + E(x)E(y) = C(x, y) + E(x)E(y), \tag{3}$$

so

$$xy - E(xy) = (\Delta x)(\Delta y) + (\Delta x)E(y) + (\Delta y)E(x) - C(x, y). \tag{4}$$

Square and take expectations to find

$$\begin{aligned} V(xy) = {} & E^2(x)V(y) + E^2(y)V(x) + E[(\Delta x)^2(\Delta y)^2] \\ & + 2E(x)E[(\Delta x)(\Delta y)^2] + 2E(y)E[(\Delta x)^2(\Delta y)] \\ & + 2E(x)E(y)C(x, y) - C^2(x, y), \end{aligned} \tag{5}$$

since $V(y) = E(\Delta y)^2$ and $V(x) = E(\Delta x)^2$. This is the result obtained by Goodman ([3], p. 712, equation (18)). Although his derivation drew on the assumption that $E(x)$ and $E(y)$ are non-zero, it must be apparent from Goodman's ([3], p. 709) remark that "some of the results . . . do not require this assumption" and from the form of equation (18) that his result was (correctly) intended to apply even if $E(x) = 0$ and/or $E(y) = 0$.

* Work on this paper was in part supported by grants from the Office of Education (OE-5-10-292) and from the Graduate School of the University of Wisconsin. We are grateful also to the referees and editor for instructive criticisms of earlier versions, the first of which was distributed in April, 1968 as Social Systems Research Institute Workshop Paper SFM 6810.

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Some specializations of the general formula (5) are of interest. If x and y are bivariate normally distributed then third moments vanish and $E[(\Delta x)^2(\Delta y)^2] = V(x)V(y)+2C^2(x, y)$—see Anderson ([1], p. 39)—whence (5) reduces to

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + 2E(x)E(y)C(x, y) + V(x)V(y) + C^2(x, y). \tag{6}$$

If x and y are uncorrelated so that $C(x, y)=0$, then (5) reduces to

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + E[(\Delta x)^2(\Delta y)^2] + 2E(x)E[(\Delta x)(\Delta y)^2] + 2E(y)E[(\Delta x)^2(\Delta y)]. \tag{7}$$

A somewhat stronger form of independence is "expectation-independence." We say that y is expectation-independent of x if and only if the conditional expectation of y, $E(y|x)$, is the same for all values of x—see Goldberger ([2], pp. 10–11) and Lord and Novick ([7], pp. 225–227) where the same concept is introduced in different terminology. In that event, $E(y|x)=E(y)$ for all x, so that $E(\Delta y|x)=0$ for all x, whence

$$E[(\Delta x)^r(\Delta y)] = E_x\{E[(\Delta x)^r(\Delta y) \mid x]\} = E_x[(\Delta x)^r E(\Delta y \mid x)] = 0$$

for every integer r. As consequences, we have not only uncorrelatedness, $C(x, y)=E[(\Delta x)(\Delta y)]=0$ when y is expectation-independent of x, but also $E[(\Delta x)^2(\Delta y)]=0$. We conclude that under this condition (5) specializes to

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + E[(\Delta x)^2(\Delta y)^2] + 2E(x)E[(\Delta x)(\Delta y)^2]. \tag{8}$$

Now suppose that y is variance-independent (i.e. homoskedastic), as well as expectation-independent, of x. With $V(y|x)=\text{constant}$ for all x, and $E(y|x)=E(y)$ for all x, it follows that

$$V(y) = V_x[E(y \mid x)] + E_x[V(y \mid x)] = V(y \mid x) = E[(\Delta y)^2 \mid x] \quad \text{for all } x,$$

whence

$$E[(\Delta x)^r(\Delta y)^2] = E_x\{E[(\Delta x)^r(\Delta y)^2 \mid x]\} = E_x\{(\Delta x)^r E[(\Delta y)^2 \mid x]\} = V(y)E(\Delta x)^r$$

for every integer r. As consequences, we have

$$E[(\Delta x)(\Delta y)^2] = V(y)E(\Delta x) = 0 \quad \text{and}$$
$$E[(\Delta x)^2(\Delta y)^2] = V(y)E[(\Delta x)^2] = V(y)V(x).$$

Inserting these into (8) we conclude that if y is expectation- and variance-independent of x, then (5) specializes to

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + V(x)V(y). \tag{9}$$

Equation (9) is identical with Goodman's ([3], p. 709) equation (2), which was established under the assumption that x and y were stochastically independent. We see that expectation- and variance-independence of either variable with respect to the other suffice to produce the same simplification for $V(xy)$ as does stochastic independence. A considerable weakening of assumptions is involved: stochastic independence of course requires that *all* conditional mo-

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ments of *both* variables, not merely two conditional moments of one variable, be constant.

Actually, even the conjunction of expectation- and variance-independence is unnecessary. A necessary and sufficient condition for $V(xy)$ to reduce to (9) is obtained by equating to zero the difference between the right-hand sides of our (5) and (9). The referee who called our attention to this pointed out that the same device could have been used on Goodman's (18) and (2).

2. THE COVARIANCE OF PRODUCTS

We now turn to the main subject of the paper. Let x, y, u, and v be jointly distributed random variables. Consider the two products xy and uv; by definition their covariance is

$$C(xy, uv) = E[xy - E(xy)][uv - E(uv)]. \tag{10}$$

Let $\Delta x = x - E(x)$, $\Delta y = y - E(y)$, $\Delta u = u - E(u)$, and $\Delta v = v - E(v)$. Multiply the expression for $xy - E(xy)$ of (4) by the corresponding expression for $uv - E(uv)$, and take expectations. Typical terms in the product include $(\Delta x)(\Delta u)E(y)E(v)$ whose expectation is $C(x, u)E(y)E(v)$, and $(\Delta x)E(y)C(u, v)$ whose expectation is 0. The result is

$$\begin{aligned} C(xy, uv) = {} & E(x)E(u)C(y, v) + E(x)E(v)C(y, u) + E(y)E(u)C(x, v) \\ & + E(y)E(v)C(x, u) + E[(\Delta x)(\Delta y)(\Delta u)(\Delta v)] \\ & + E(x)E[(\Delta y)(\Delta u)(\Delta v)] + E(y)E[(\Delta x)(\Delta u)(\Delta v)] \\ & + E(u)E[(\Delta x)(\Delta y)(\Delta v)] + E(v)E[(\Delta x)(\Delta y)(\Delta u)] \\ & - C(x, y)C(u, v). \end{aligned} \tag{11}$$

This is our formula for the covariance of products of random variables. It may be specialized in a variety of ways, as several examples should suffice to suggest. If we set $x = u$ and $y = v$, then (11) reduces to the variance formula (5), as it should since $C(xy, xy) = V(xy)$. If we set $u = 1$ so that $E(u) = 1$ and $\Delta u = 0$, then (11) yields

$$C(xy, v) = E(x)C(y, v) + E(y)C(x, v) + E[(\Delta x)(\Delta y)(\Delta v)]. \tag{12}$$

Under multivariate normality all third moments vanish, while $E[(\Delta x)(\Delta y)(\Delta u)(\Delta v)] = C(x, y)C(u, v) + C(x, u)C(y, v) + C(x, v)C(y, u)$—see Anderson ([1], p. 39)—in which case (11) reduces to

$$\begin{aligned} C(xy, uv) = {} & E(x)E(u)C(y, v) + E(x)E(v)C(y, u) + E(y)E(u)C(x, v) \\ & + E(y)E(v)C(x, u) + C(x, u)C(y, v) + C(x, v)C(y, u). \end{aligned} \tag{13}$$

We note that the conventional asymptotic approximation procedure—see Kendall and Stuart ([6], p. 232)—would yield for the covariance of xy with uv,

$$\begin{aligned} C^*(xy, uv) = {} & E(x)E(u)C(y, v) + E(x)E(v)C(y, u) + E(y)E(u)C(x, v) \\ & + E(y)E(v)C(x, u), \end{aligned} \tag{14}$$

where $C^*(. , .)$ denotes an approximate covariance. Note that (14) is the sum of the first four terms on the right of (11), so that the error involved in using the

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approximation $C^*(xy, uv)$ rather than the exact $C(xy, uv)$ is the sum of the last six terms in (11). Under multivariate normality, the error is that given by the sum of the last two terms in (13).

The consequences of various degrees of independence in terms of simplifying $C(xy, uv)$ can be obtained by the methods of Section 2. We cite only one: If the pair (x, y) is expectation-independent and covariance-independent of the pair (u, v), then (11) reduces to $C(xy, uv)=0$.

3. DISCUSSION

Product variables occur naturally in regression contexts to capture non-additive (i.e., interaction) effects. An examination of the results above makes it clear that zero-order correlations involving product variables may not be scale-free in the sense of being invariant with respect to linear transformations of the underlying variables. For, variances and covariances of product variables involve the expectations as well as the central moments of the underlying variables.

Product variables may also arise in classical test score theory, if the product of two scores happens to be the relevant variable. The reliability of a variable is defined as the correlation between two parallel measurements on it; under classical assumptions this reduces to the ratio of the variance of the true variable to the (ordinary) variance of its measures—see Gulliksen ([5], pp. 13–14). Under similar assumptions, the reliability of a product variable will reduce to a similar ratio of variances. Again, our results will make it clear that the reliability of the product may not be scale-free with respect to linear transformations of its components.

In a second paper, Goodman [4] extended his analysis to cover the variance of the product of K variables, and a similar extension for the covariance might be desirable here. However, for such higher-dimension problems the enumeration involved is tedious. A more efficient scheme, which works with symmetric functions and partitions, is given in Kendall and Stuart ([6], Chapter 12). Indeed some readers may find that scheme more attractive even for our low-dimension problem.

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