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Making Sense Out of Consensus

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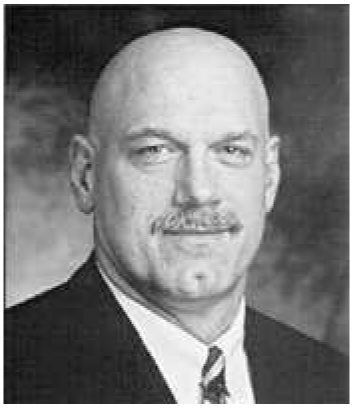

How did Minnesota governor Jesse Ventura defeat the major-party candidates in 1998? Moving beyond some of the common explanations, voting theorists take a hard look at plurality-based election systems---"the only procedure," according to Donald Saari, "that will elect someone who's despised by almost two-thirds of the voters."

"How did that guy win?"

The morning after Election Day 1998, political commentators all over the country were scratching their heads at the result of the Minnesota governor's race. Jesse "The Body" Ventura, third-party candidate and former pro wrestler, had just beaten two establishment candidates---Republican Norm Coleman and Democrat Hubert Humphrey III---who were far ahead of him in the polls.

The theories abounded: Ventura had won the young male vote. Ventura had won because Minnesota allows voters to register on Election Day. Ventura had won because the winters are long in Minnesota and voters need some way to keep themselves amused.

According to Donald Saari of the University of California, Irvine (formerly of Northwestern University), there's one theory all the political commentators missed: Ventura won because of the way we count our votes. The problem may have been the plurality-based election system, which Saari calls "the only procedure that will elect someone who's despised by almost two-thirds of the voters."

Because we are so used to the plurality vote, it may not occur to most people that there are other ways to run an election. [For a glimpse of the Italian system, see Ivar Stakgold's interview with Mario Primicerio, former mayor of Florence.] In fact, dozens of ways have been proposed, and several have been used in practice. The most familiar alternative is a runoff election, in which the top two candidates in the popular vote go against each other head-to-head. Here are some of the others:

- The Borda count. Each voter ranks all the candidates. If there are n candidates, the top one on the list receives *n* points, the second receives (*n* - 1) points, and so on. For the tally, the points from all the voters are added up. (An equivalent way to compute the Borda count: Run a round-robin "tournament," with each candidate going up against all others in one-on-one races. Then add the total votes for each candidate.) The Borda count is commonly used in college sports polls.

- The Condorcet criterion. The Marquis de Condorcet---the founder of voting theory---suggested in 1785 that the winner of an election should be the candidate who defeats all the others in a head-to-head match-up. Because a "Condorcet winner" does not emerge in every election, however, a backup procedure is needed.

- Approval voting. Each voter casts one vote for every candidate he or she approves of. This method has been adopted by six scientific societies, including the Mathematical Association of America and the American Mathematical Society.

- Single transferable vote. As in the Borda count, each voter submits a complete ranking of the candidates. The first-place votes are tallied by the plurality method, and the last-place candidate is discarded. The voters who ranked that candidate first then have their votes "transferred" to their second-place candidate, and the ballots are retallied. This procedure is repeated until a majority winner emerges. This method can be adapted to parliamentary or legislative elections in which more than one winner is chosen, and is currently used in Australia, Ireland, and Cambridge, Massachusetts.

- Dictatorship. In a dictatorship---perhaps the procedure with the longest history of all---one voter controls the outcome of the election (although other voters may be allowed to express their preferences). Besides its use in various governments throughout history, it also arises in corporations in which one investor owns more than 50% of the stock.

The Mathematical Soul of Arrow’s Theorem

The bad news is that, even with all these systems to choose from, there’s not one that is perfect. The different systems not only disagree with each other, they are almost all susceptible to a variety of paradoxes. In multistage systems, such as runoffs or the single transferable vote, it can be advantageous not to vote for your favorite candidate in the first round. In almost all systems, the dropping out of a candidate---even a candidate who is in last place---can totally scramble the rankings of the other candidates. In 1950, Stanford economist Kenneth Arrow (who later won a Nobel Prize for his work) proved that such paradoxes are inevitable under any voting method other than a dictatorship.

More precisely, Arrow proved that there is no way (except for a dictatorship) to construct a "social choice function" that satisfies two apparently innocuous conditions: First, if every voter prefers candidate A to candidate B, then the social choice function should also rank candidate A over candidate B (the "Pareto condition"). Second, if the social choice function ranks candidate A over candidate B, then it should continue to do so regardless of whether any voter changes his or her opinion of candidate C. (In particular, the entry or exit of candidate C from the race can be considered a drastic way of making voters revise their rating of that candidate.) Arrow called his second condition "independence of irrelevant alternatives."

Fifty years later, the paradox discovered by Arrow continues to befuddle people who know nothing of his theorem. A fascinating example occurred in the 1995 world figure-skating championships, when then-newcomer Michelle Kwan skated well enough to finish fourth in the final standings---and, more importantly, to reverse the standings of two skaters who had long since finished, originally placing second and third. Naturally, the commentators tut-tutted: Why should the standings of skaters be affected by someone who finished below them? After a similar incident in the men’s European championships in 1997, the International Skating Union adopted a new scoring system that was intended to prevent such "flip-flops." Yet as any voting theorist could have pointed out, a voting system based on ordinal rankings-as both the old and the new ISU systems are-will always be vulnerable to flip-flops, according to Arrow’s theorem.

But perhaps we shouldn’t be too hard on the ISU. Arrow’s theorem was startling even to mathematicians. When Saari first heard of it, he recalls, "I said it couldn’t possibly be true." For fifteen years, he has been trying to understand the mathematical soul of Arrow’s theorem, and now he believes he’s got it. What’s more, he says, he can now identify "all possible paradoxes" in voting theory, and thereby tell which method is least likely to elect a wildly unpopular candidate. In two 50-page articles in this January’s issue of Economic Theory, he comes to a surprising---and still controversial---conclusion: The choice of a voting method may depend on the circumstances, but most of the time there is a best method, and it’s not the plurality vote. It’s the Borda count.

Illogical Behavior or Simple Mathematics?

To analyze the strengths and weaknesses of various voting methods, mathematicians generally do not look at the results of actual elections. Real-world elections are messy things, and polls often fail to reveal the true motivations of the voters. Ever since Condorcet, mathematicians (including Arrow and Saari) have also made an assumption that might be debatable in the real world: that individual voters are rational. In other words, if they prefer A to B and B to C, they also prefer A to C. If so, then every voter can, in theory, put together a ranking of his or her preferences (possibly with ties). The collection of all such rankings is the electorate's "voter profile." Voter profiles make it easy to tell what would happen under different voting methods, as in an example that Saari likes to use in his lectures:

> Fifteen people were deciding what beverage to serve at a party---wine, beer, or milk. Unbeknownst to them, their voter profile was as follows: Six preferred milk first, wine second, and beer third. Five preferred beer first, wine second, and milk third; four preferred wine first, beer second, and milk third. At first, the group tried to decide on a beverage by a simple show of hands---in other words, a plurality vote. Naturally, milk was the winner. But immediately, a ruckus ensued. After all, milk was the last choice of 60% of the voters!
>
> To arrive at a fairer result, some of the party-goers suggested a runoff election. The two top vote-getters in the show of hands, milk and beer, were pitted against each other. All the wine lovers switched their votes to beer, and so beer won, 9-6. Everyone was happy until someone asked, "Wait a minute! Why are we getting beer, when 10 of us would rather have wine?" In fact, under the Condorcet criterion and the Borda count, wine would be the winner. Rumor has it that the party never took place, because the organizers were too busy arguing over the beverage.

"The moral," according to Saari, "is that the outcome of an election may depend on the procedure for counting the votes, not the wishes of the electorate." Mathematician William Zwicker of Union College, who has taught voting theory to freshmen several times, says that examples like this one always surprise his students. "It's not Arrow's theorem that shakes them up. What does shake them up is the fact that there is more than one possible voting system, and they have competing virtues."

Perhaps even more unsettling to the students would be the first, and in some ways the most fundamental, voting paradox of all, discovered by Condorcet in 1785: A society made up of rational people can vote irrationally.

The proof, once again, is based on the construction of a voter profile. Condorcet's profile was very simple. Suppose three voters are choosing their favorite beverage. Voter 1 likes milk best, wine second, and beer third. Voter 2 likes wine best, beer second, and milk third. Voter 3 likes beer best, milk second, and wine third. Now think about what happens if you offer these three people a choice of milk or wine. By a 2-1 vote, they would choose milk. Next, suppose you offer them a choice of wine or beer. Again by a 2-1 vote, they would take wine.

It seems as if the group preference is milk first, wine second, and beer third, right? And yet, if you offer these same three people a choice between milk and beer, they would vote for beer. This is the sort of illogical behavior that, if it happened in an actual election, would have political scientists shaking their heads and talking about "voter rebellions." But there's nothing more to it than simple math.

Linear Algebra Applied to Voter Profiles

The reason Condorcet's paradox seems so paradoxical, Saari says, is that we are breaking the symmetry of the voter profile by looking at the pairwise elections. In fact, Condorcet's profile has what mathematicians call "cyclic symmetry." (One voter ranks A > B > C, one voter ranks B > C > A, and one ranks C > A > B. Each of these orderings is a cyclic permutation of the others.) A voter profile can also have "reversal symmetry"; an example is a two-voter election in which one voter ranks A > B > C and the other C > B > A.

According to Saari's analysis, your choice of a voting method boils down to the way you want to treat profiles of these two types. If you think that the outcome for a profile with cyclic symmetry should be a tie (and everyone questioned by Saari has replied that it should), then you should choose a voting method based on rankings-in any such system, the candidates in a cyclic profile have the same number of firsts, seconds, and so forth. If you believe that the result for a profile with reversal symmetry should be a tie, then you should choose a voting method based on pairwise comparisons, like Condorcet's criterion.

And if you think both of the above profiles should be ties? In that case you have one choice, the only voting method that can be expressed both as a ranking method and as a pairwise-competition method: the Borda count.

Saari's proof of this fact is a tour de force of linear algebra, applied to the vector space of all voter profiles. In an n-candidate election, there are n-factorial different rankings of the candidates. A single profile gives the number of voters who subscribe to each ranking, and hence the profile space is a vector space with n-factorial dimensions.

Large as they are, Saari showed that these vector spaces have a very straightforward decomposition into subspaces. First, he discovered an enormous subspace, called "the kernel," that has no effect at all on the outcome of any method that involves pairwise comparisons of candidates or rankings.

A smaller subspace, called the "basic component," does affect the outcome of elections, but it affects every ranking or pairwise method identically. In essence, this is the space of profiles in which society behaves with perfect rationality, so there is nothing for the different voting methods to disagree over.

All the action---all the voting paradoxes, and all the disagreements between methods---thus boils down to the orthogonal component of the kernel and basic subspaces. This component, Saari proved, is spanned by the profiles with cyclic symmetry and the profiles with reversal symmetry. The Borda count is the unique voting method that treats all such profiles as ties.

So far, the main criticism of Saari's work focuses not on the mathematics but on its interpretation. Although impressed by "some terrific mathematics," Zwicker objects that Saari's work limits the universe of possible voting methods to those based on rankings or pairwise comparisons. In this way it ignores one of the Borda count's most important competitors, approval voting.

In fact, approval voting is familiar to many mathematicians because of its adoption by AMS and MAA. In 1988, Saari, one of the approval method's earliest critics, debated its effectiveness in the pages of Public Choice with Steven Brams, a political scientist at New York University who is one of the method's most outspoken advocates. Saari called approval voting a "cure worse than the disease," on the grounds that it divorces the results of an election from the voters' preferences. That is, the same voter profile can produce many different results, depending on where each voter decides to draw the line between approved and non-approved candidates. Brams, however, viewed this as an advantage, because it gives each voter "sovereignty" over the way she expresses her preferences.

California "Debacle"

The debate between the Borda count and approval voting is not likely to be resolved soon. Ironically, in a field dedicated to helping society reach consensus, voting theorists themselves are far from reaching a consensus. The one thing that Saari and Brams do agree on, and quite emphatically, is that the election system now used in the U.S. is worse than either of them. We may not notice it when there are only two strong candidates, but as soon as a third one enters the picture, strange things begin to happen.

Consider, for example, how John McCain would have done in this year's California primary if it had been conducted by a Borda count. A Sacramento Bee exit poll showed that California voters would have voted for McCain in a two-candidate race against Gore, 48-43. The same voters would have given Gore the nod in a two-candidate race against Bush, 51-43. The Bee did not ask how the voters would have voted in a two-candidate race of McCain against Bush, because such a scenario could not happen under our system. But a look at the official voting totals showed that Republicans split 60-35 in favor of Bush, while the Democrats who voted Republican split 64-31 in favor of McCain. If we assume the entire Democratic party would have split that way, then the hypothetical McCain-Bush race then comes out 50-45 in McCain's favor. And the Borda count gives:

McCain 48	Gore 43	
McCain 50		Bush 45
	Gore 51	Bush 43
McCain 98	Gore 94	Bush 88

The California primary was universally considered a debacle for McCain, and he withdrew from the race two days later. The polls cannot prove, of course, that he would have won the election, but shouldn't he have had the chance?

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