

Poisson Distribution

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1 Introduction

The Poisson distribution is used to represent integer-valued counts, such as the number of events observed during a period of time or the number of things observed in a given amount of space.

The Poisson distribution is single-peaked—it has only one mode. However, its shape can change dramatically. If the mode is 0, the Poisson’s probability mass function can look like a “ski slope.” On the other hand, if the mode is some large value, then the Poisson distribution has a distribution which is symmetrical (to the eye, it is almost Normal in appearance⁰). The Poisson is a discrete distribution, and so it only approximates a continuous Normal to a limited extent.

Typically used in event count data to show, for example the number of protests in London each day for the month of July.

2 Mathematical Description of the Poisson Distribution

The Poisson has a single parameter, which is customarily known as λ (The Greek letter lambda).

The Poisson probability mass function says the probability that there are exactly x occurrences (x has to be a non-negative integer) in a time-interval is:

$$f(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}$$

where e is Euler’s constant, the base of the natural logarithm (approximately 2.71828...), and $x!$ is the factorial of x ($x! = x \cdot (x - 1) \cdot \dots \cdot 2 \cdot 1$).

The Poisson distribution has a singular distinction that its expected value is equal to its variance, and both of them are equal to λ . Formally,

$$E(x) = \lambda$$

$$Var(x) = \lambda$$

The parameter λ is a positive real number, equal to the expected number of occurrences that occur during the given interval. For instance, if the events occur on average every 4 minutes, and you are interested in the number of events occurring in a 10 minute interval, you would use as model a Poisson distribution with $\lambda = 2.5$. “

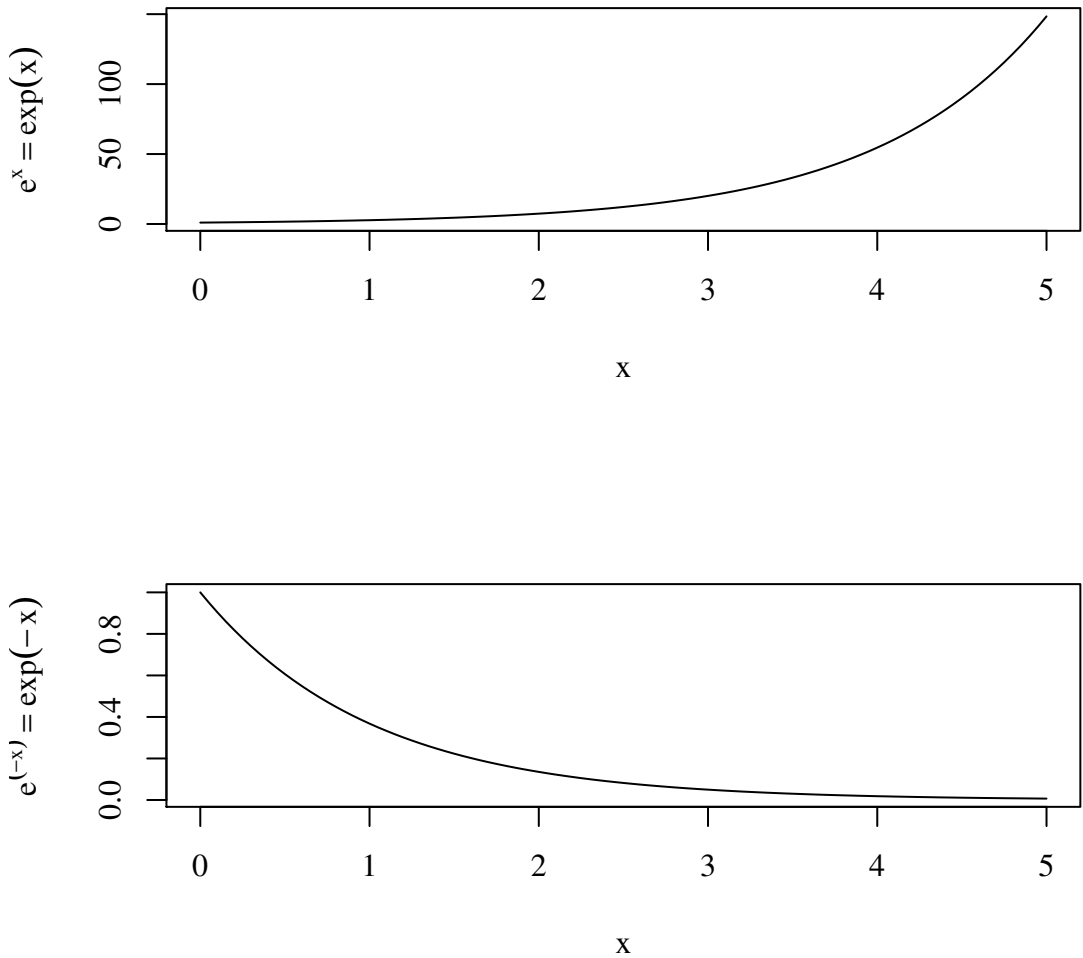
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The parameter lambda can be estimated by dividing the average number of successes observed over a certain amount of time, \bar{x} , by the length of the time interval, t .

$$\hat{\lambda} = \frac{\bar{x}}{t}$$

¹http://en.wikipedia.org/wiki/Poisson_distribution

Figure 1: The Exponential Function



2.1 Other Moments

The skewness is

$$Skewness(x) = \lambda^{-1/2}$$

The Kurtosis is

$$Kurtosis(x) = \lambda$$

3 Mathematical background

3.1 Euler's constant

Poisson uses the exponential function with base of the natural log, e known as Euler's (pronounced in English as "Oiler's") constant. Figure 1 shows the shape of the functions e^x , which is also known as $\exp(x)$, and e^{-x} (or $\exp(-x)$).

3.2 Poisson as an Infinite Sum

Recall that the cumulative distribution—the sum of probabilities of all possible outcomes—must equal 1.

Begin with a “guess” that the probability mass function might be

$$\frac{\lambda^x}{x!}$$

The values that this implies are presented in this table:

x	0	1	2	3	4	5	...		∞
$\lambda^x/x!$	1	λ^1	$\lambda^2/2!$	$\lambda^3/3!$	$\lambda^4/4!$	$\lambda^5/5!$			$\lambda^\infty/\infty!$

The sum of the items in the second row would be

$$1 + \lambda + \lambda^2/2! + \lambda^3/3! + \lambda^4/4! + \lambda^5/5! + \dots + \lambda^\infty/\infty! \tag{1}$$

which is one definition of the value of $exp(\lambda)$. The reader can check that differentiating the sum in expression 1 simply gives back exactly the same sum, one piece of evidence that this really is $exp(\lambda)$.

In order to turn the values shown in the table into probabilities, then, it is only necessary to divide by $exp(\lambda)$, because doing so makes the sum of all probabilities equal to one. Hence, if we started with the kernel of the probability mass function, $\lambda^x/x!$, and divide by $exp(\lambda)$, we have a probability mass function

$$\frac{\lambda^x}{exp(\lambda)x!} = \frac{exp(-\lambda)\lambda^x}{x!}$$

4 Illustrations

The appearance of the probability mass function changes radically as λ is changed. As λ is increased beyond 10, the appearance is rather similar to a Normal density.

The Poisson is a discrete distribution, and to emphasize that fact, we indicate the probability of each value of x by a bar with a small dot on top. This kind of representation is used because it helps the reader to remember that the values being considered are discrete integers (the Poisson is not defined for values such as 1.1).

In Figure 2, the probabilities for several small values of λ are presented. Note that as long as $\lambda < 1$, the expected value is less than 1 and the mode is 0.

In Figure 3, the tendency of the Poisson to approach a symmetric, single-peaked distribution is illustrated.

If we choose to play games with the reader, we can make the Poisson look more and more like a Normal distribution. The same data is plotted in Figures 4 and 5. The only difference is that the probability values are connected by lines. In Figure 4, the line is a “step” function, which is closer to maintaining the meaning of the discrete distribution, while Figure 5 uses R’s default line to “smooth” out the kinks between points. A set of plots created with a smoothed line appears very Normal indeed.

Figure 2: Poisson with small λ values

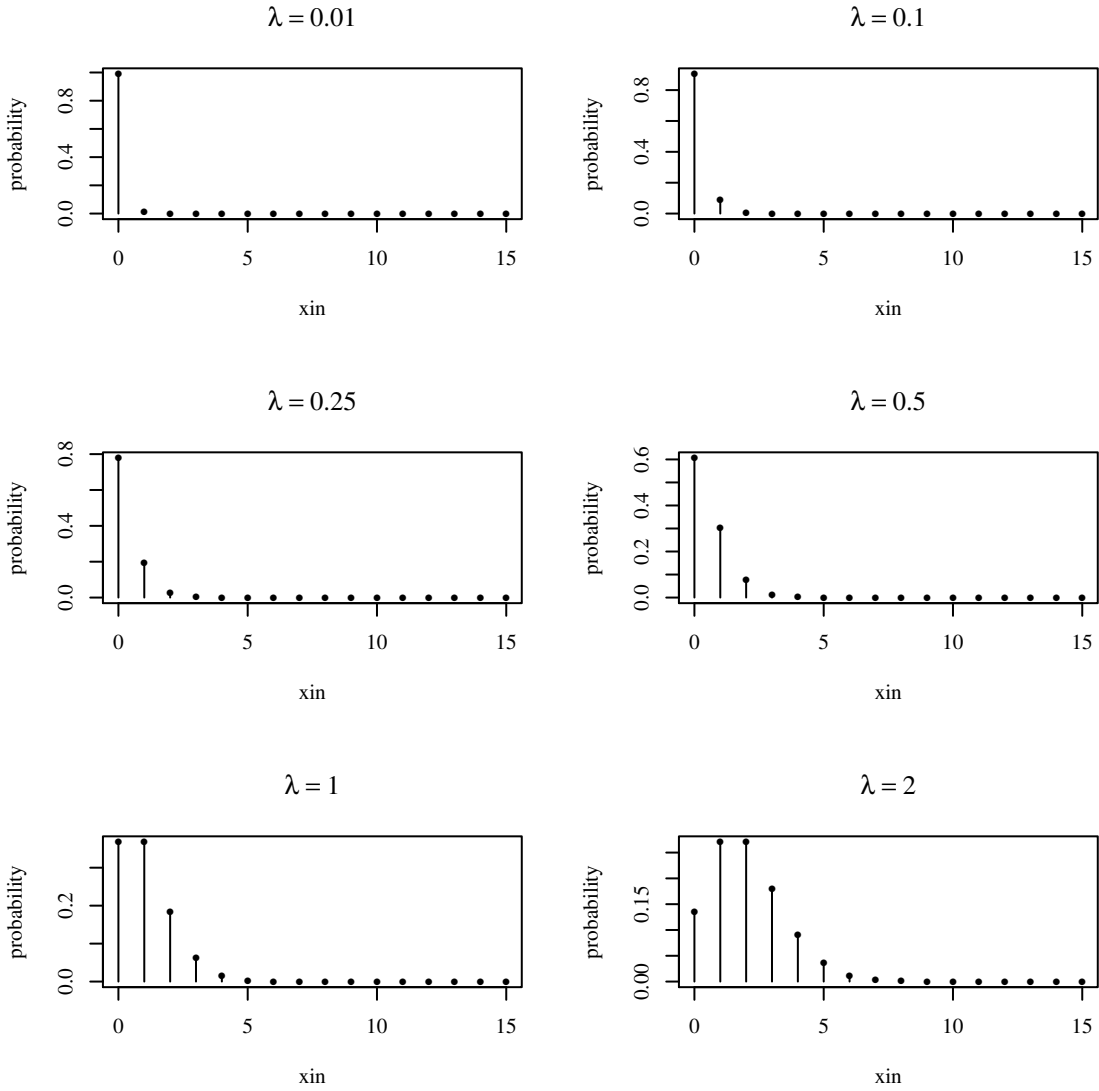


Figure 3: Poisson with larger λ values

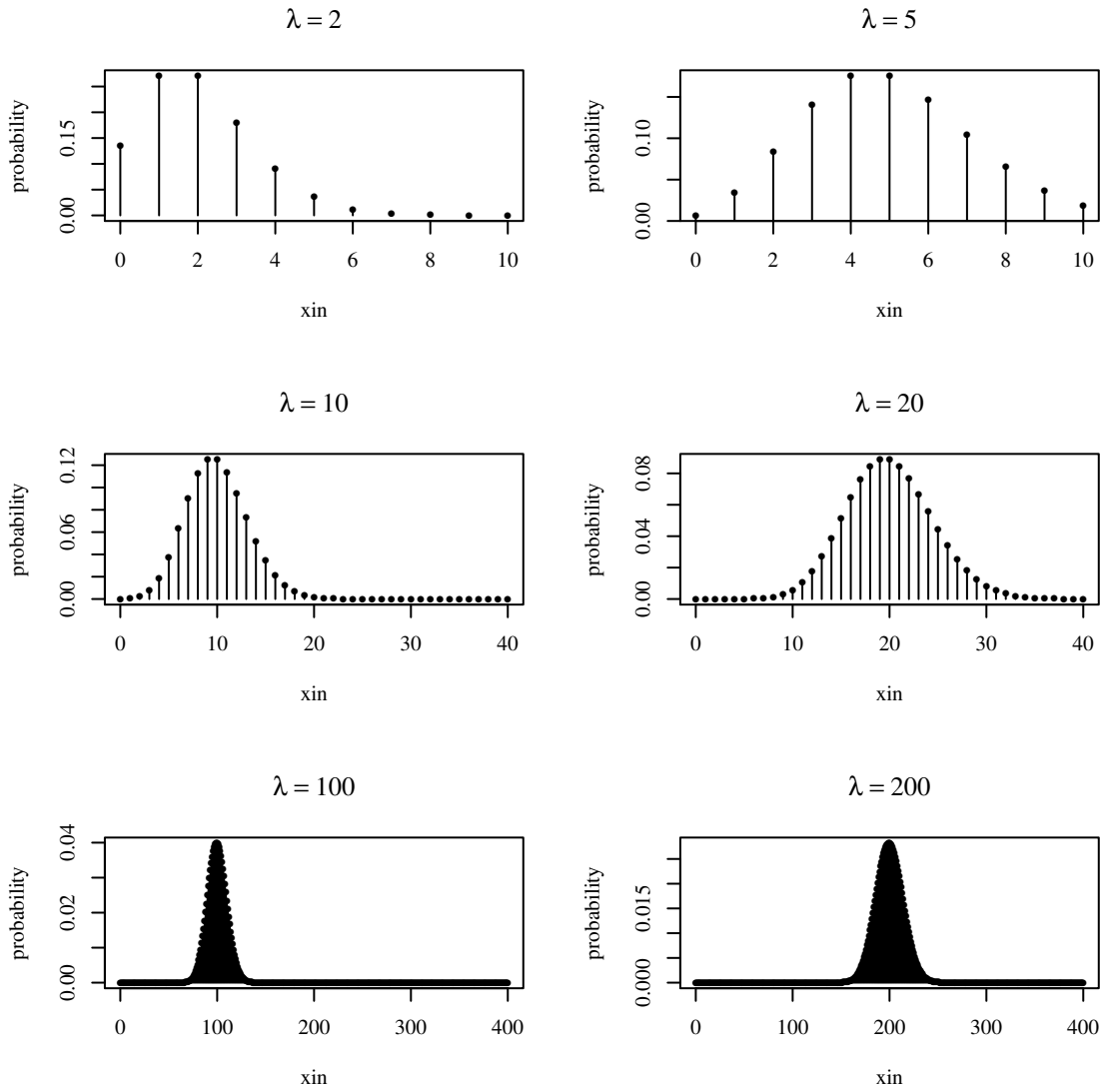


Figure 4: Stair Step Illustration of a Poisson

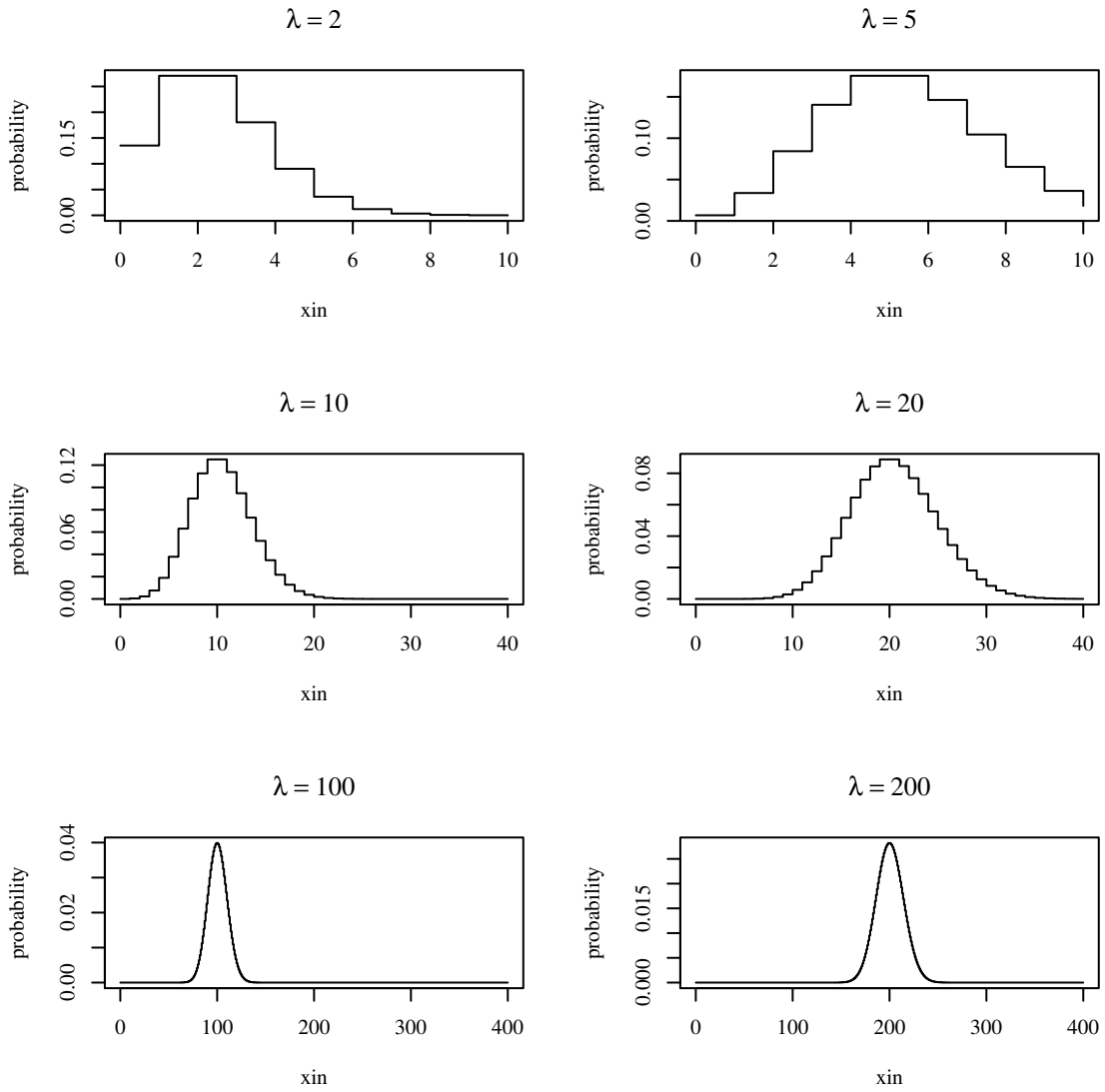


Figure 5: Smooth Lines Create Illusion of Normality

