Pareto Distribution

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1 Introduction

The Pareto distribution is a probability model for continuous variables. It looks like a "ski slope." The Pareto distribution is commonly used with socio-economic and other naturally occuring quantities that are distributed in a "ski slope" manner with very long right tails. The long right tails describe inequality, the possibility of a few extremely large outcomes. The distribution appears to be primarily used in the business and economics fields, but also within political science. (For example, Jones and Baumgartner's The Politics of Attention (2005) considers the possibility the policy changes are distributed as a Pareto distribution). The Pareto distribution works best in situations when we want to understand the long right tails. The discrete counterpart of the Pareto distribution is Zipf's law, which is sometimes refered to as the "zeta distribution."

2 Mathematical Definition

The Pareto's density function has 2 primary parameters, shape and location (known as scale in some treatments).

The location parameter sets the position of the "left edge" of the probability density. The only outcomes that can be observed are greater than the value of the location parameter. It is required that $location > 0$.

The shape parameter determines the steepness of the "ski slope."

If x_i is Pareto distributed, the probability density function is:

$$f(x_i) = \begin{cases} 0 & \leq \ location \\ \left(\frac{shape}{location}\right)\left(\frac{location}{x_i}\right)^{(zhape+1)} & > \ location \end{cases} \tag{1}$$

It is sometimes rearranged like so:

$$f(x_i) = \begin{cases} 0 & \leq \ location \\ \frac{shape \cdot location^{shape}}{x_i^{(shape+1)}} & > \ location \end{cases} \quad (2)$$

In order to have finite moments, the Pareto must have a shape that is greater than 2 (see variance below). Anything less than or equal to 2 results in a variance that is undefined.

3 Illustrations

The probability density function of the Pareto distribution changes when one puts in different values of the parameters. Consider the following R code, which can be used to create the illustration of a Pareto density function with location and scale equal to 1 in Figure 1.

```
library(VGAM)
xvals <- seq(0.1,10,len=1000)
plot(xvals, dpareto(xvals, location=1, shape=1), type="l", xlab="
   possible values of x", ylab="probability of x", main="Pareto
   Probability Density")
text(2, .6, "shape=1, location=1", pos=4, col=1)
```

In order to illustrate the impact of changing the shape parameter, let's set the location parameter equal to 1. Examples of Pareto density function for several values of the shape parameter (with a fixed location) are found in Figure 2. The effect of changing the shape parameter is seen in the height of the distribution.

We can explore the effect of changing the location parameter by repeating the previous exercise, fixing the shape at 1 and varying the location. Consider Figure 3, which results from this code:

```
leftedge <- c(1,3,5)
x1 <- seq(leftedge[1],20,len=1000)
y1 <- dpareto(x1, location = leftedge[1], shape = 1)
x2 <- seq(leftedge[2], 20, len = 1000)
y2 <- dpareto(x2, location = leftedge[2], shape = 1)
x3 <- seq(leftedge[3], 20, len = 1000)
y3 <- dpareto(x3, location = leftedge[3], shape = 1)
plot(x1, y1, type="l", xlab = "possible values of x", ylab = "
   probability of x", ylim = c(0,1))
lines(x2, y2, lty = 2, col=2)
lines(x3, y3, lty = 3, col=4)
legend("topright", c("location=1","location=3","location=5"), lty
   =1:3, col=c(1,2,4))
```

As illustrated in Figure 3, the location represents the left boundary for the Pareto distribution. Changes in the location simply shift this boundary to the left or right (recall that the model requires $location > 0$). This Figure demonstrates that the appearance of the distribution changes when the location parameter changes, even if shape is kept fixed.

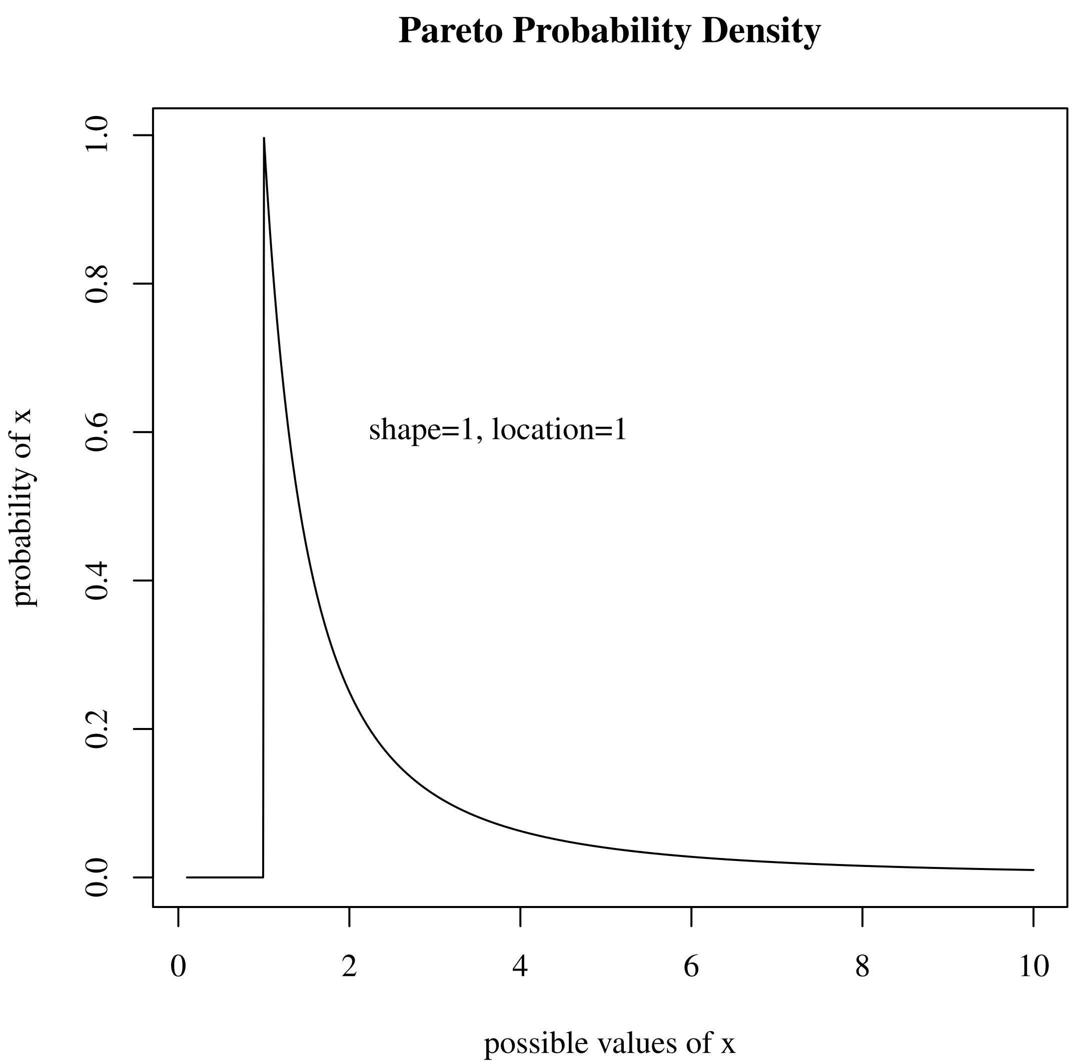

Figure 1: Pareto Density Function

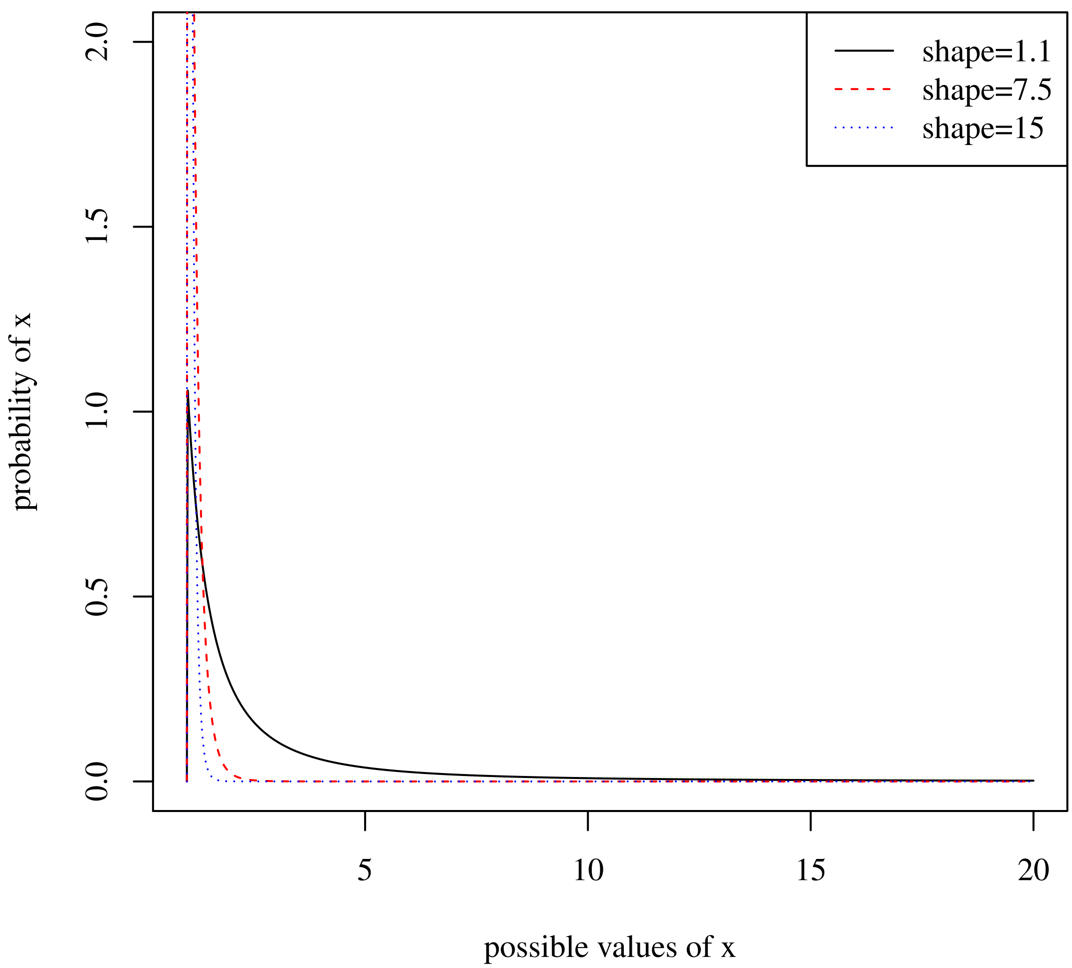

Figure 2: Pareto Density with Various Shapes

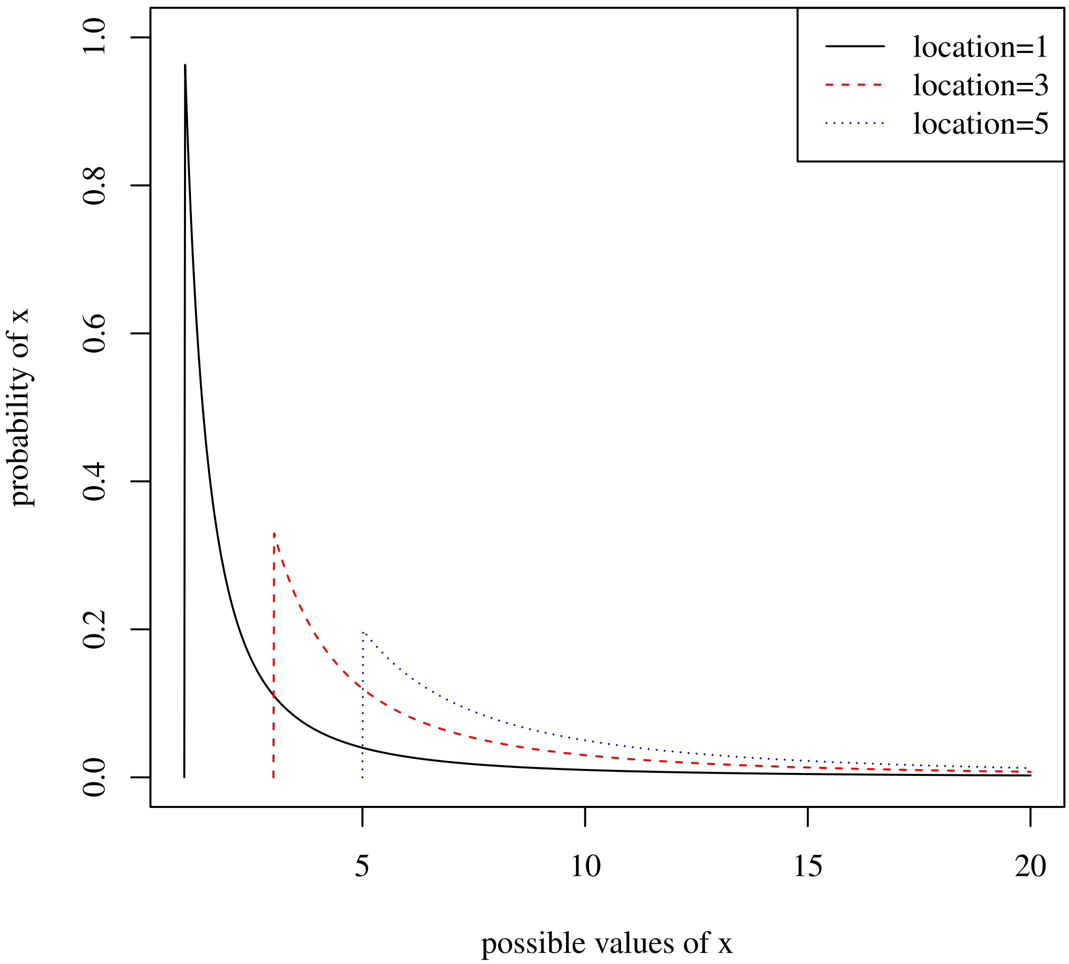

Figure 3: Pareto Density with Various Locations

4 Expected Value, Variance, and the role of the parameters

The Pareto probability distribution has these interesting properties:

$$E\left(x_i\right) = \frac{shape \cdot location}{shape - 1} \qquad 1 < shape$$

If the shape is less than or equal to 1, the expected value of a Pareto is infinite, or undefined.

$$Var\left(x_i\right) = \frac{shape \cdot (location)^2}{(shape-1)^2 (shape-2)} \; 2 < shape$$

If the shape is less than or equal to 2, the variance of a Pareto is infinite, or undefined.

$$Std.Dev(x) = \frac{shape \cdot location}{(shape-1)\sqrt{shape-2}} \qquad 2 < shape$$

With a Pareto distribution, the mode is always equal to the value of the *location* parameter. The height of the density function at the mode is equal to the *shape* parameter divided by the *location*.

$$max\, f\left(x_i\right) = \frac{shape}{location}$$

If x_i has the probability density function $Pareto(shape, location)$, then $y_i = \frac{1}{x_i}$ has the density function $f\left(y_i\right) = (shape \cdot location)^{shape} y_i^{(shape-1)}$. This distribution is called the power function distribution. Its moments are simply the negative moments of the Pareto distribution.[1]

[1] http://www.xycoon.com/par_relationships1.htm