

Laplace Distribution

Raminta Stockute and Paul Johnson

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1 Introduction

This is a continuous probability distribution. It is named after a French mathematician. Wikipedia points out that it is also known as a double exponential distribution, because it reminds one of an exponential distribution “spliced together back-to-back.”

The most outstanding characteristics of this distribution are that it is unimodal and symmetric.

2 Mathematical definition

This distribution is characterized by location θ (any real number) and scale λ (has to be greater than a 0) parameters.

The probability density function of *Laplace*(θ, λ) is:

$$f(x|\theta, \lambda) = \frac{1}{2\lambda} \exp\left(-\frac{|x-\theta|}{\lambda}\right).$$

The cumulative density function looks even more impressive, yet rather easy to integrate because of the absolute value in the formula:

$$F(x|\theta, \lambda) = \frac{1}{2} \exp\left(-\frac{|x-\theta|}{\lambda}\right), \text{ when } (x \leq \theta).$$

and

$$F(x|\theta, \lambda) = 1 - \frac{1}{2} \exp\left(-\frac{|\theta-x|}{\lambda}\right), \text{ when } (x > \theta).$$

The exponential distribution's probability density is defined for $x > 0$

$$\textit{Exponential}(\lambda) : f(x|\lambda) = \frac{1}{\lambda} \exp(-x/\lambda), x > 0$$

Unlike the exponential, the Laplace is defined $-\infty < x < \infty$.

If $\theta = 0$, then the probability density function for Laplace on $x > 0$ is equal to $1/2$ of the probability of the exponential. In Figure 1, we illustrate this fact by plotting the probability density of the Laplace on $(-15, 15)$ side-by-side with an exponential distribution, and then in the figure below, one can observe that if the exponential is divided by half, then it is equal to the Laplace. The R code which generates the figure is:

```
par(mfrow=c(2, 2))
x1 <- seq(-15, 15, by = 0.05)
mylaplace1 <- dlaplace(x1, location = 0, scale = 1)
plot(x1, mylaplace1, type = "l", xlab = "x", ylab = "P(x)",
     main="Laplace, location=0, scale=1")
x2 <- seq(0, 15, by = 0.05)
myexp1 <- dexp(x2, rate = 1)
plot(x2, myexp1, type = "l", xlab = "x", ylab = "P(x)", main = "
     Exponential, rate=1")
myexp2 <- 0.5*dexp(x1, rate=1)
plot(x1, myexp2, type = "l", xlab = "x", ylab = "0.5*P(x)", main
     = "0.5*Exponential PDF")
```

```
library(VGAM)
```

```
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```

3 Moments

The expected value of a Laplace distribution is

$$E(x) = \theta$$

As in the case of other symmetrical distributions, such as the Normal and the logistic distributions, Laplace's location is the same as its mean, median, and mode.

The variance is:

$$Var(x) = 2\lambda^2.$$

4 Illustrations

From the Figure 2, we see that the scale parameter determines the width of the distribution. From Figure 3, it is apparent that changing the location simply shifts the probability density curve to the right or to the left.

The Laplace can be compared against the Normal distribution. Recall that a Normal distribution $N(\mu, \sigma^2)$ has an expected value of μ and a variance equal to σ^2 . Suppose we fix the mean of a Normal to equal the mean of a Laplace distribution, and then also match the variances of the two. In Figure 4, we compare $N(4, 8)$ and $L(4, 2)$, both of which have variance equal to 8. The distributions appear to be quite similar, except that Laplace has higher spike and slightly thicker tails .

```
library (VGAM)
```

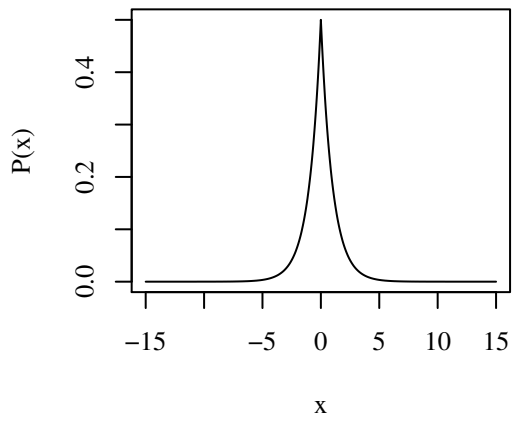
```
library (VGAM)
```

5 Conclusion

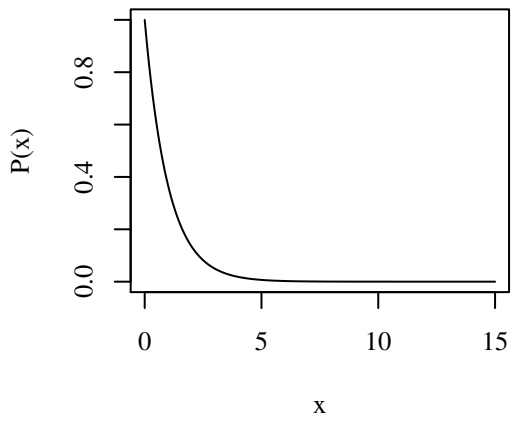
If we have real-valued observations, the errors can be distributed either in normal or in Laplace. Let us take gene expression data as an example. A distribution of gene expression errors tends to be in Laplace form. Better yet, if the distribution is a bit asymmetric, there is an Asymmetric Laplace to allow for this asymmetry.

Figure 1: Laplace and Exponential

Laplace, location=0, scale=1



Exponential, rate=1



0.5*Exponential PDF

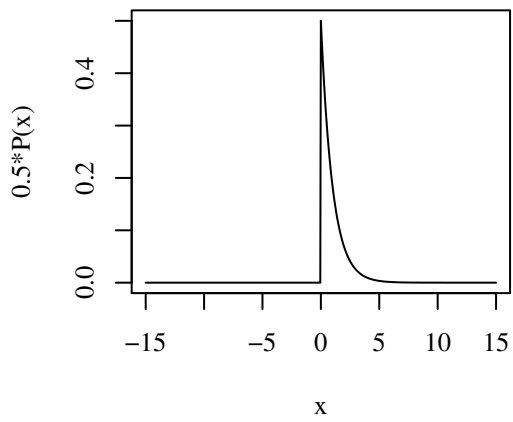


Figure 2: Laplace Density Function

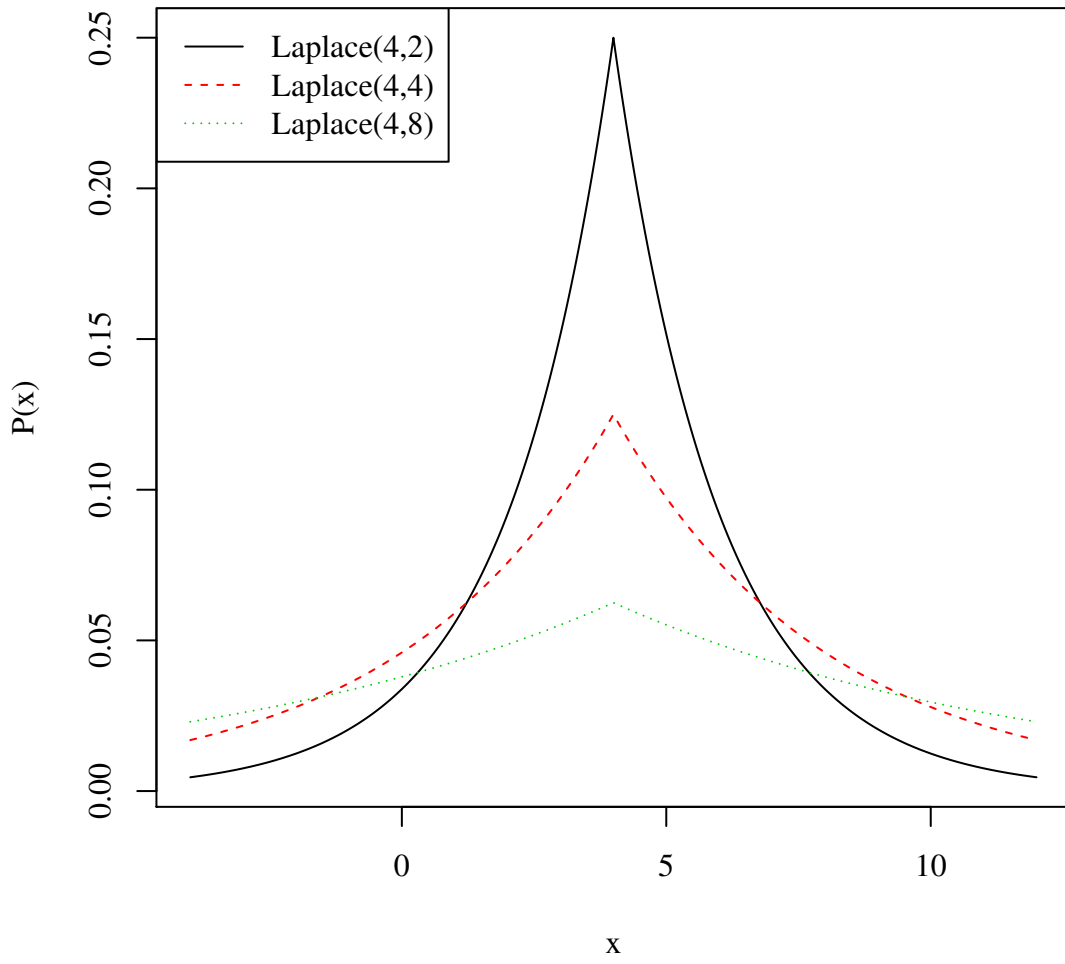


Figure 3: Impact of location change on Laplace Density Function

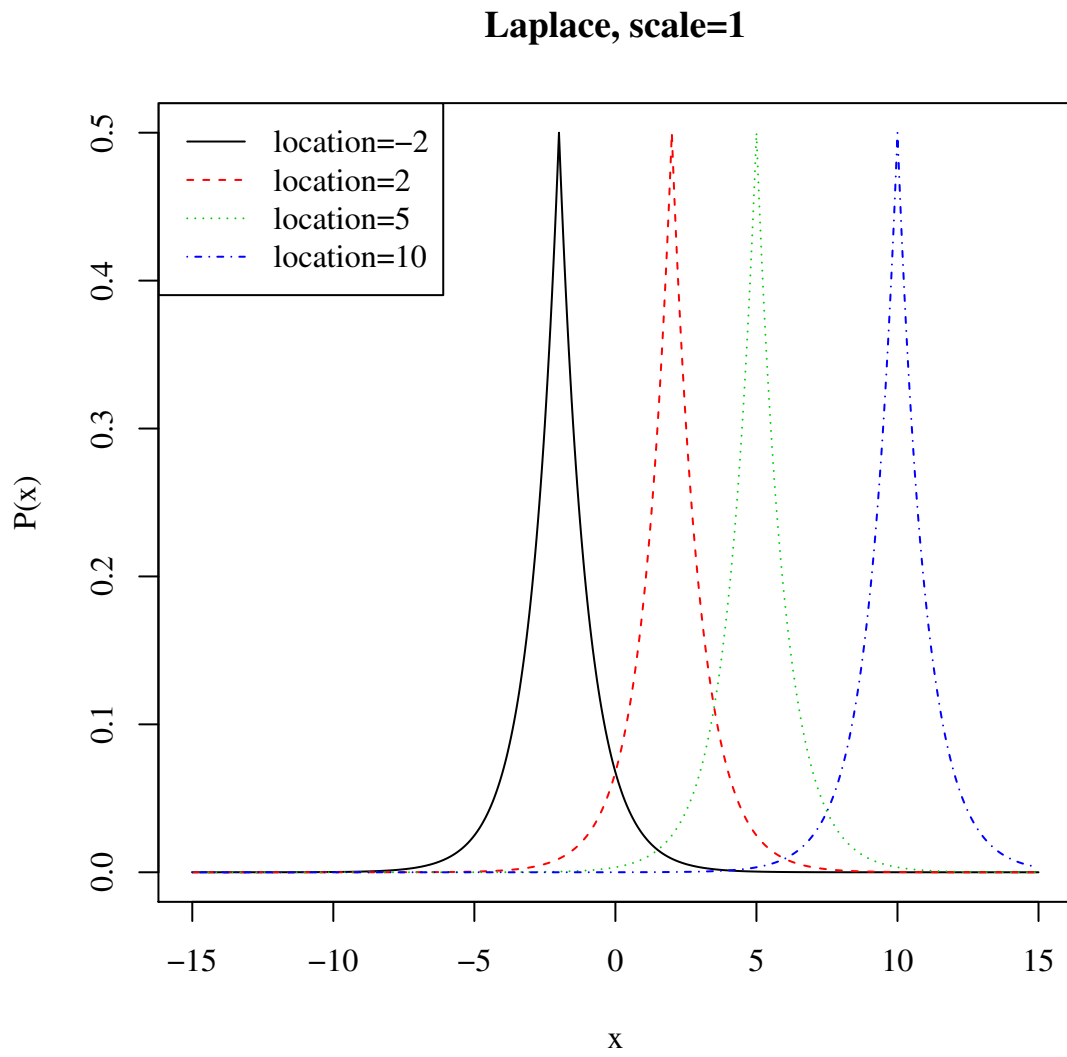
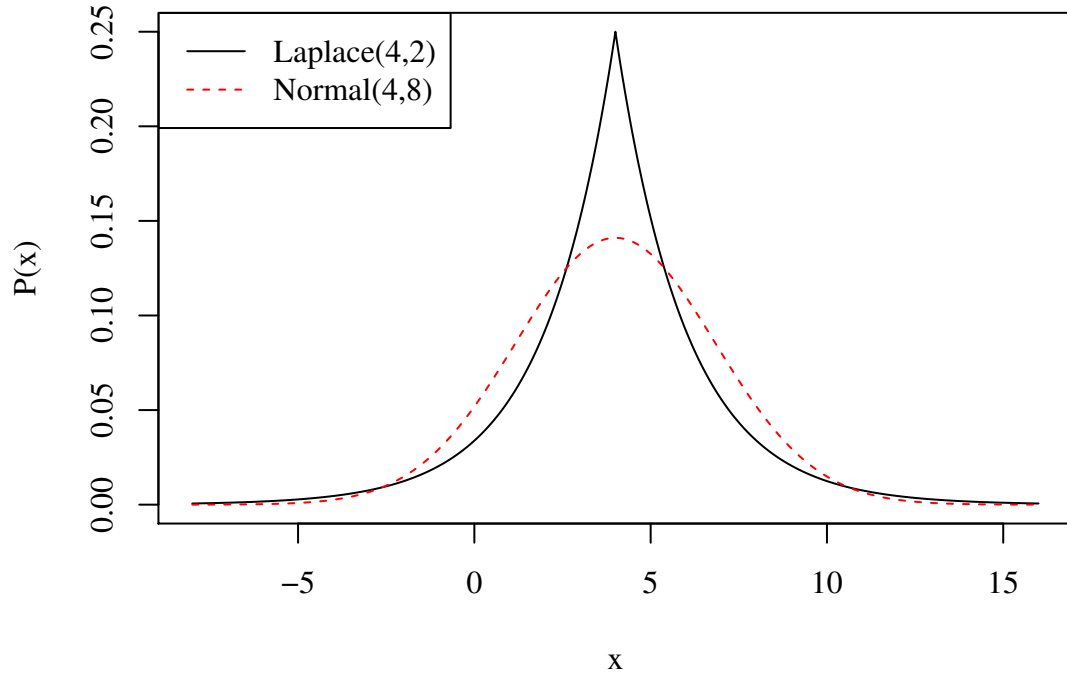


Figure 4: Laplace and Normal Densities



Expanded view, $x > 8$

